Pythagorean-hodograph curves in Euclidean spaces of dimension greater than 3

Takis Sakkalis
Mathematics Laboratory, Agricultural University of Athens,
75 Iera Odos, Athens 11855, GREECE

Rida T. Farouki*
Department of Mechanical and Aerospace Engineering,
University of California, Davis, CA 95616, USA

Abstract

A polynomial Pythagorean–hodograph (PH) curve $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ in $\mathbb{R}^n$ is characterized by the property that its derivative components satisfy the Pythagorean condition $x_1'^2(t) + \cdots + x_n'^2(t) = \sigma^2(t)$ for some polynomial $\sigma(t)$, ensuring that the arc length $s(t) = \int \sigma(t)\,\mathrm{d}t$ is simply a polynomial in the curve parameter t. PH curves have thus far been extensively studied in $\mathbb{R}^2$ and $\mathbb{R}^3$, by means of the complex–number and the quaternion or Hopf map representations, and the basic theory and algorithms for their practical construction and analysis are currently well–developed. However, the case of PH curves in $\mathbb{R}^n$ for $n > 3$ remains largely unexplored, due to difficulties with the characterization of Pythagorean $(n+1)$–tuples when $n > 3$. Invoking recent results from number theory, we characterize the structure of PH curves in dimensions $n = 5$ and $n = 9$, and investigate some of their properties.

Keywords: Pythagorean–hodograph curves; Parameterization of n-tuples; Complex numbers; Quaternions; Octonions; Hopf map.

e–mail: stp@aua.gr, farouki@ucdavis.edu

*Corresponding author

1 Introduction

A polynomial curve $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ in $\mathbb{R}^n$ is a *Pythagorean–hodograph* (PH) *curve* if its hodograph (derivative) $\mathbf{r}'(t) = (x_1'(t), \ldots, x_n'(t))$ has components that satisfy, for some polynomial $\sigma(t)$, the Pythagorean condition

$$x_1'^2(t) \,+\, \cdots \,+\, x_n'^2(t) \;=\; \sigma^2(t)\,. \tag{1}$$

Consequently, the parametric speed $|\mathbf{r}'(t)| = \mathrm{d}s/\mathrm{d}t$ — i.e., the rate of change of arc length s with respect to the parameter t — is just a *polynomial* (rather than the square root of a polynomial) in t. This feature endows PH curves with many advantageous algorithms for geometric design, motion control, animation, path planning, and related applications.

PH curves were first introduced [10] in $\mathbb{R}^2$ using a simple characterization for Pythagorean polynomial triples, that can be conveniently expressed in terms of a *complex–variable* model [3, 8, 9]. Subsequently, PH curves in $\mathbb{R}^3$ were characterized in terms of the *quaternion* and *Hopf map* models [2, 6] — which facilitate the identification of some interesting special sub–classes [5, 7, 13]. A further extension of the theory involves study of the PH condition (1) under the Minkowski (rather than Euclidean) metric [14] — i.e., with the $+$ sign before $x_n'^2(t)$ in (1) replaced by a $-$ sign. The study of solutions to $x_1'^2(t)+x_2'^2(t)-x_3'^2(t) = \sigma^2(t)$ in the Minkowski space $\mathbb{R}^{(2,1)}$ is motivated by the desire to exactly reconstruct the boundary of a planar domain from its *medial axis transform*. Further details on all these different Euclidean and Minkowski PH curve incarnations may be found in [4].

This paper shows that the Hopf model, used to characterize PH curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, can also be generalized to obtain characterizations for PH curves in $\mathbb{R}^5$ and $\mathbb{R}^9$. The outline for the remainder of the paper is as follows. Section 2 reviews some basic results concerning Pythagorean $(n+1)$–tuples of real polynomials, and uses them to extend the theory of planar and spatial PH curves to $\mathbb{R}^5$ and $\mathbb{R}^9$ (and thereby also $\mathbb{R}^4$ and $\mathbb{R}^8$). A one–to–one correspondence between the regular PH curves in $\mathbb{R}^n$ and (equivalence classes of) regular "ordinary" polynomial curves in $\mathbb{R}^{2n-2}$ is also established. Section 3, whose results are independent from the rest of the paper, provides a characterization of the conditions under which two complex, quaternion, or octonion polynomials are constant multiples of each other. These results are then used in Section 4 to obtain simple criteria identifying linear and planar instances of higher–dimensional PH curves. Finally, Section 5 summarizes the key results of this study, and identifies some open problems.

2 Pythagorean $(n+1)$-tuples

We first review some recent results from [15] that are essential to the subsequent analysis. The construction of PH curves has thus far been based on the solution of the Pythagorean condition (1) in the cases $n = 2$ and $n = 3$. To construct PH curves in $\mathbb{R}^n$ when $n > 3$, it is essential to characterize all solutions of (1) for $n > 3$.

Let $\mathbb{R}[t]$ be the ring of univariate real polynomials, and let $n \geq 1$. Then a *Pythagorean* $(n+1)$*–tuple* over $\mathbb{R}[t]$ is a vector $(p_1(t), \ldots, p_n(t), p_{n+1}(t)) \in \mathbb{R}^{n+1}[t]$, such that

$$p_1^2(t) + \cdots + p_n^2(t) \;=\; p_{n+1}^2(t)\,. \tag{2}$$

Theorem 4.1 on page 1268 of [15], which uses ideas from quadratic forms, gives all solutions of (2) for some special values of n. Before stating it, we need to introduce some notation. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ denote the sets of real numbers, complex numbers, quaternions, and octonions, where we make the identifications $\mathbb{C} = \mathbb{R}^2$, $\mathbb{H} = \mathbb{R}^4$, and $\mathbb{O} = \mathbb{R}^8$.

Now if $\mathcal{A}$ is a member of one of these sets, we denote by $\|\mathcal{A}\|$ its norm, by $\mathcal{A}^*$ its conjugate (such that $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = \|\mathcal{A}\|^2$), and by $\mathrm{Re}(\mathcal{A})$ and $\mathrm{Im}(\mathcal{A})$ its real and imaginary parts. Also, the appropriate (real, complex, quaternion, or octonion) product is imputed whenever two such elements appear in juxtaposition. See the Appendix for a brief review of these ideas in the case of the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

Theorem 2.1 *Let $\mathcal{A}(t), \mathcal{B}(t) \in \mathbb{R}^{n-1}[t]$ and $c(t) \in \mathbb{R}[t]$. Then, modulo permutations, the solutions of* (2) *over $\mathbb{R}[t]$ in the cases $n = 2, 3, 5, 9$ are of the form*

$$(p_1, p_2, \ldots, p_n, p_{n+1}) \;=\; c\,(\|\mathcal{A}\|^2 - \|\mathcal{B}\|^2, 2\mathcal{A}\,\mathcal{B}^*,\, \|\mathcal{A}\|^2 + \|\mathcal{B}\|^2)\,. \tag{3}$$

Now for $k = 1, 2, 4, 8$ let R_k denote the set $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and let $R_{k+1} = \mathbb{R} \times R_k$. A closer look at (3) reveals that the solution set of (2) is an instance of the real *Hopf fibration* $H : R_k \times R_k \to R_{k+1}$ defined for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in R_k$ by

$$H(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\|\boldsymbol{\alpha}\|^2 - \|\boldsymbol{\beta}\|^2, 2\,\boldsymbol{\alpha}\,\boldsymbol{\beta}^*)\,. \tag{4}$$

Indeed, if R_k and R_{k+1} are replaced by the corresponding polynomial rings $R_k[t]$ and $R_{k+1}[t]$ we note that $c\,(\|\mathcal{A}\|^2 - \|\mathcal{B}\|^2, 2\mathcal{A}\,\mathcal{B}^*, \|\mathcal{A}\|^2 + \|\mathcal{B}\|^2) = c\,(H(\mathcal{A}, \mathcal{B}), \|H(\mathcal{A}, \mathcal{B})\|)$.

2.1 Pythagorean–hodograph curves in $\mathbb{R}^5$ and $\mathbb{R}^9$

In view of (3) we are now able to completely characterize PH curves in dimensions $n = 2, 3, 5$ and 9. Henceforth, unless otherwise stated, n will be $2, 3, 5$ or 9.

Let $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ be a Pythagorean–hodograph curve in $\mathbb{R}^n$. Then there exist polynomials $\mathcal{A}(t), \mathcal{B}(t) \in R_{n-1}[t]$ such that

$$\mathbf{r}'(t) \,=\, H(\mathcal{A}(t), \mathcal{B}(t)) \qquad \text{and} \qquad \|\mathbf{r}'(t)\|^2 \,=\, \|H(\mathcal{A}(t), \mathcal{B}(t))\|^2 \,. \tag{5}$$

Since we wish to restrict our attention to "regular" PH curves, for which the components of $\mathbf{r}'(t)$ have no non–constant common factor, we may take $\gcd(\mathcal{A}(t), \mathcal{B}(t)) = 1$ in (5) — this is a necessary (but not sufficient) condition for a regular PH curve.

We now delineate (5) for each of the above dimensions. First, we re–write $H(\mathcal{A}(t), \mathcal{B}(t))$ in terms of the familiar *dot* $(\cdot)$ and *cross* $(\times)$ products. Namely, for $\mathcal{A}(t), \mathcal{B}(t) \in R_{n-1}[t]$ we set $\mathcal{A}(t) = (\alpha_0(t), \boldsymbol{\alpha}(t))$, $\mathcal{B}(t) = (\beta_0(t), \boldsymbol{\beta}(t))$ where $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t) \in R_{n-2}[t]$ — i.e., $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are null or in $\mathbb{R}[t]$, $\mathbb{R}^3[t]$, or $\mathbb{R}^7[t]$. We recall a result of Massey [12] stating that the only spaces $\mathbb{R}^m$ that admit a (non–degenerate) cross product are $\mathbb{R}^3$ and $\mathbb{R}^7$ (the dot product is defined in $\mathbb{R}^m$ for any $m \geq 1$). A simple calculation then shows that (5) can be written as

$$H(\mathcal{A}, \mathcal{B}) \,=\, (\|\mathcal{A}\|^2 - \|\mathcal{B}\|^2, 2\,\mathcal{A} \cdot \mathcal{B}, 2(\beta_0\,\boldsymbol{\alpha} - \alpha_0\,\boldsymbol{\beta} - \boldsymbol{\alpha} \times \boldsymbol{\beta}))\,. \tag{6}$$

Case 2D: Here $\mathbf{r}(t) = (x(t), y(t))$ and $\mathbf{r}' = (a^2 - b^2, 2ab)$ where $a, b \in \mathbb{R}[t]$, $\gcd(a, b) = 1$.

Case 3D: Here $\mathbf{r}(t) = (x(t), y(t), z(t))$ and if $\mathcal{A} = u + \mathrm{i}\,v, \mathcal{B} = q + \mathrm{i}\,p \in \mathbb{R}^2[t]$ we have

$$\mathbf{r}'(t) \,=\, (\|\mathcal{A}\|^2 - \|\mathcal{B}\|^2, 2\,\mathcal{A}\,\mathcal{B}^*) \,=\, (u^2 + v^2 - p^2 - q^2, 2(uq + vp), 2(vq - up))$$

with $\gcd(u, v, p, q) = 1$.

Case 5D: Now $\mathbf{r}(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ and if $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$, $\mathcal{B}(t) = a(t) + \mathbf{i}\,b(t) + \mathbf{j}\,c(t) + \mathbf{k}\,d(t) \in \mathbb{R}^4[t]$, we have

$$\begin{aligned}
x_1' &= u^2 + v^2 + p^2 + q^2 - a^2 - b^2 - c^2 - d^2\,, \\
x_2' &= 2(ua + vb + pc + qd)\,, \\
x_3' &= 2(va - ub + qc - pd)\,, \\
x_4' &= 2(pa - uc + vd - qb)\,, \\
x_5' &= 2(qa - ud + pb - vc)\,.
\end{aligned}$$

Case 9D: Finally, $\mathbf{r}(t) = (x_1(t), x_2(t), \cdots, x_9(t))$ and if $\mathcal{A}(t) = a_0 + a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 + a_4\,\mathbf{e}_4 + a_5\,\mathbf{e}_5 + a_6\,\mathbf{e}_6 + a_7\,\mathbf{e}_7$, $\mathcal{B}(t) = b_0 + b_1\,\mathbf{e}_1 + b_2\,\mathbf{e}_2 + b_3\,\mathbf{e}_3 + b_4\,\mathbf{e}_4 + b_5\,\mathbf{e}_5 + b_6\,\mathbf{e}_6 + b_7\,\mathbf{e}_7 \in \mathbb{R}^8[t]$ then

$$
\begin{aligned}
x_1' &= (a_0^2 + a_1^2 + \cdots + a_7^2) - (b_0^2 + b_1^2 + \cdots + b_7^2)\,,\\
x_2' &= 2(a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 + a_7b_7)\,,\\
x_3' &= 2(a_1b_0 - a_0b_1 + a_4b_2 - a_2b_4 + a_6b_5 - a_5b_6 + a_7b_3 - a_3b_7)\,,\\
x_4' &= 2(a_2b_0 - a_0b_2 + a_5b_3 - a_3b_5 + a_7b_6 - a_6b_7 + a_1b_4 - a_4b_1)\,,\\
x_5' &= 2(a_3b_0 - a_0b_3 + a_6b_4 - a_4b_6 + a_1b_7 - a_7b_1 + a_2b_5 - a_5b_2)\,,\\
x_6' &= 2(a_4b_0 - a_0b_4 + a_7b_5 - a_5b_7 + a_2b_1 - a_1b_2 + a_3b_6 - a_6b_3)\,,\\
x_7' &= 2(a_5b_0 - a_0b_5 + a_1b_6 - a_6b_1 + a_3b_2 - a_2b_3 + a_4b_7 - a_7b_4)\,,\\
x_8' &= 2(a_6b_0 - a_0b_6 + a_2b_7 - a_7b_2 + a_4b_3 - a_3b_4 + a_5b_1 - a_1b_5)\,,\\
x_9' &= 2(a_7b_0 - a_0b_7 + a_3b_1 - a_1b_3 + a_5b_4 - a_4b_5 + a_6b_2 - a_2b_6)\,.
\end{aligned}
$$

Remark 2.1 *The above results can also be used to construct characterizations of PH curves in $\mathbb{R}^4$ and $\mathbb{R}^8$. Suppose we set $x_5' = 2(qa - ud + pb - vc) = 0$ in* Case 5D*. The solutions of this equation can be parameterized as*

$$
\begin{aligned}
(a, -d, b, -c) \;=\; & z_1(u, -q, 0, 0) \;+\; z_2(p, 0, -q, 0) \;+\; z_3(v, 0, 0, -q)\\
& +\; z_4(0, p, -u, 0) \;+\; z_5(0, v, 0, -u) \;+\; z_6(0, 0, v, -p)\,. \qquad (7)
\end{aligned}
$$

Hence, we obtain a parameterization for PH curves in $\mathbb{R}^4$ in terms of 10 parameters, namely $u, v, p, q, z_1, \ldots, z_6$. By analogous arguments in Case 9D*, a parameterization for PH curves in $\mathbb{R}^8$ can be obtained in terms of 36 parameters.*

2.2 Hopf map revisited

Proposition 3.1 on page 368 of [3] shows that there are "as many" 2D regular PH curves as 2D regular polynomial curves, and in Section 22.5, page 483 of [4] the question is raised as to whether a similar result is true for spatial curves. We now investigate the question of correspondence between PH curves and "ordinary" polynomial curves, using the Hopf map characterization for PH curves.

Let n be $2, 3, 5$ or 9, and let $\widehat{\Pi}_n$ denote the set of all regular PH curves in $\mathbb{R}^n$, and Π_{2n-2} the set of all regular "ordinary" polynomial curves in $\mathbb{R}^{2n-2}$. We define an equivalence relation $\sim$ on Π_{2n-2} as follows. Let $\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^{2n-2}$ be a regular polynomial curve, and write

$\mathbf{r}(t) = (r_1(t), \cdots, r_{n-1}(t), r_n(t), \cdots, r_{2n-2}(t)) = (\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$, where $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t) \in R_{n-1}[t]$. Then if $\mathbf{s}(t) = (s_1(t), \cdots, s_{n-1}(t), s_n(t), \cdots, s_{2n-2}(t)) = (\boldsymbol{\gamma}(t), \boldsymbol{\delta}(t))$ we say that $\mathbf{r}(t) \sim \mathbf{s}(t)$ if a unit element $\boldsymbol{\lambda} \in R_{n-1}$ exists, such that $(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) = (\boldsymbol{\gamma}(t)\,\boldsymbol{\lambda}, \boldsymbol{\delta}(t)\,\boldsymbol{\lambda})$. Let $\Pi_{2n-2}/\!\sim$ be the set of equivalence classes, and denote one of its elements by $[\mathbf{r}(t)]$. Then we have the following result.

Proposition 2.1 *There is a bijection between the sets* $\Pi_{2n-2}/\!\sim$ *and* $\widehat{\Pi}_n$.

To prove this, we make use of the following remark based on the description of the fibers of the Hopf map H. As is well–known, these fibers are simply the unit spheres of appropriate dimension. For convenience, we give here a brief proof of this remarkable fact.

Remark 2.2 *Let* H *be the Hopf fibration defined by*

$$H \,:\, R_{n-1}[t] \times R_{n-1}[t] \to \mathbb{R}[t] \times R_{n-1}[t]\,, \quad H(\boldsymbol{\alpha}, \boldsymbol{\beta}) \,=\, (\|\boldsymbol{\alpha}\|^2 - \|\boldsymbol{\beta}\|^2, 2\,\boldsymbol{\alpha}\,\boldsymbol{\beta}^*)\,.$$

Then $H^{-1}(\|\boldsymbol{\alpha}\|^2 - \|\boldsymbol{\beta}\|^2, 2\,\boldsymbol{\alpha}\,\boldsymbol{\beta}^*) = \{\, (\boldsymbol{\gamma}, \boldsymbol{\delta}) \mid \boldsymbol{\alpha} = \boldsymbol{\gamma}\,\boldsymbol{\lambda},\ \boldsymbol{\beta} = \boldsymbol{\delta}\,\boldsymbol{\lambda} \,\}$ *where* $\boldsymbol{\lambda} \in S^{n-2}$.

Proof: First observe that $H^{-1}(0,0) = (0,0)$. Now suppose that $H(\boldsymbol{\alpha}, \boldsymbol{\beta}) = H(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Then if $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} = 0$, we have either $\boldsymbol{\gamma} = 0$ or $\boldsymbol{\delta} = 0$, and since $\|\boldsymbol{\alpha}\|^2 = \|\boldsymbol{\gamma}\|^2 - \|\boldsymbol{\delta}\|^2$, we see that $\boldsymbol{\delta} = 0$. Since $R_{n-1}[t]$ is a division ring and $\boldsymbol{\gamma} \neq 0$, there exists precisely one element $\boldsymbol{\lambda} \in R_{n-1}(t)$ with $\boldsymbol{\alpha} = \boldsymbol{\gamma}\,\boldsymbol{\lambda}$. But since $\|\boldsymbol{\alpha}\|^2 = \|\boldsymbol{\gamma}\|^2$ we deduce that $\boldsymbol{\lambda} \in S^{n-2}$, as required. An analogous argument holds when $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\beta} \neq 0$.

Now assume that $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \neq 0$. Then there exists a $\boldsymbol{\lambda} \in R_{n-1}(t)$ such that $\boldsymbol{\alpha} = \boldsymbol{\gamma}\,\boldsymbol{\lambda}$. Since $2\,\boldsymbol{\alpha}\,\boldsymbol{\beta}^* = 2\,\boldsymbol{\gamma}\,\boldsymbol{\delta}^*$, we have $\boldsymbol{\delta}^* = \boldsymbol{\lambda}\,\boldsymbol{\beta}^*$, and hence $\|\boldsymbol{\alpha}\|^2 = \|\boldsymbol{\lambda}\|^2\,\|\boldsymbol{\gamma}\|^2$ and $\|\boldsymbol{\delta}\|^2 = \|\boldsymbol{\lambda}\|^2\,\|\boldsymbol{\beta}\|^2$. Then $\|\boldsymbol{\alpha}|^2 - \|\boldsymbol{\beta}\|^2 = \|\boldsymbol{\gamma}\|^2 - \|\boldsymbol{\delta}\|^2$ gives $(\|\boldsymbol{\lambda}\|^2 - 1)\,\|\boldsymbol{\gamma}\|^2 = (1 - \|\boldsymbol{\lambda}\|^2)\,\|\boldsymbol{\beta}\|^2$, and thus $\|\boldsymbol{\lambda}\| = 1$ or $\boldsymbol{\lambda} \in S^{n-2}$. Hence $\boldsymbol{\delta}^* = \boldsymbol{\lambda}\,\boldsymbol{\beta}^* \;\Rightarrow\; \boldsymbol{\delta} = \boldsymbol{\beta}\,\boldsymbol{\lambda}^*$ or $\boldsymbol{\beta} = \boldsymbol{\delta}\,\boldsymbol{\lambda}$ (since $\boldsymbol{\lambda}^*\,\boldsymbol{\lambda} = 1$), as required. ■

Proof of Proposition 2.1 For $\mathbf{r}(t) = (\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) \in \Pi_{2n-2}$ let $\mathcal{P}(\mathbf{r}(t)) = H(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) \in \widehat{\Pi}_n$. The map $\mathcal{P}$ is evidently onto. From Remark 2.2, we have $[\mathbf{r}(t)] = [\mathbf{s}(t)]$ for $\mathbf{r}(t), \mathbf{s}(t) \in \mathcal{P}^{-1}(\mathcal{P}(\mathbf{r}(t)))$. Thus, $\mathcal{P}$ induces a bijection between the sets $\Pi_{2n-2}/\!\sim$ and $\widehat{\Pi}_n$. ■

3 Linear dependence of polynomials over division rings

This section, which can be read independently from the rest of the paper, gives conditions under which proportionality relations of the form "$\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$" hold for $\mathcal{A}(t), \mathcal{B}(t) \in F[t]$

and $\mathcal{C} \in F$, where F is one of the normed rings $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. These conditions are, in turn, used in Section 4 for developing criteria under which higher–dimension PH curves specialize to linear or planar loci. They are also used to identify new "refined" conditions for linear dependency of two vectors, as well as dependency among polynomials over $\mathbb{R}[t]$.

Our first task is to identify sufficient and necessary conditions on non–zero elements $\mathcal{A}(t)$, $\mathcal{B}(t)$ of $F[t]$ such that

$$\mathcal{A}(t) \,=\, \mathcal{C}\,\mathcal{B}(t) \qquad \text{for some } \ \mathcal{C} \in F\,. \tag{8}$$

To motivate the solution, consider (8) first in the case of $\mathbb{R}[t]$. Let $a(t), b(t) \in \mathbb{R}[t]$ be such that $a(t) = \gamma\, b(t)$ for $\gamma \in \mathbb{R}$. Then we can eliminate γ by taking derivatives, $a'(t) = \gamma\, b'(t)$, to obtain $b(t)\, a'(t) = b(t)\, \gamma\, b'(t) = \gamma\, b(t)\, b'(t) = a(t)\, b'(t)$. This condition is also sufficient since $a(t)b'(t) - a'(t)b(t)$ is simply the Wronskian $W(a(t), b(t))$ of $a(t)$ and $b(t)$ [1]. The condition can also be phrased as $b^2(t)a(t)a'(t) = a^2(t)b(t)b'(t)$.

3.1 The main result

Motivated by the above, it is now easy to find a necessary condition for (8) to hold. Indeed, since $\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$ we have $\|\mathcal{A}(t)\|^2 = \|\mathcal{C}\|^2\, \|\mathcal{B}(t)\|^2$, and by differentiating and conjugating we obtain $\mathcal{A}'(t) = \mathcal{C}\,\mathcal{B}'(t)$ and $\mathcal{A}^*(t) = \mathcal{B}^*(t)\,\mathcal{C}^*$. Therefore, $\mathcal{A}^*(t)\mathcal{A}'(t) = \mathcal{B}^*(t)\,\mathcal{C}^*\mathcal{C}\,\mathcal{B}'(t)$, and hence we have

$$\|\mathcal{B}(t)\|^2 \mathcal{A}^*(t)\mathcal{A}'(t) \,=\, \|\mathcal{A}(t)\|^2\, \mathcal{B}^*(t)\mathcal{B}'(t)\,. \tag{9}$$

The following result states that this is also a sufficient condition.

Proposition 3.1 *For non–zero elements $\mathcal{A}(t)$, $\mathcal{B}(t)$ of $F[t]$, we have $\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$ for some $\mathcal{C} \in F$ if and only if $\|\mathcal{B}(t)\|^2\mathcal{A}^*(t)\mathcal{A}'(t) = \|\mathcal{A}(t)\|^2\mathcal{B}^*(t)\mathcal{B}'(t)$.*

Proof: Omitting the dependence of $\mathcal{A}(t)$, $\mathcal{B}(t)$ on t for brevity, we set $\mathcal{B}^{-1} = \|\mathcal{B}\|^{-2}\mathcal{B}^*$ and then have

$$(\mathcal{A}\,\mathcal{B}^{-1})' \,=\, (\mathcal{A}\|\mathcal{B}\|^{-2}\mathcal{B}^*)' \,=\, \mathcal{A}'\,\|\mathcal{B}\|^{-2}\mathcal{B}^* \,-\, \mathcal{A}\,(\mathcal{B}'^*\mathcal{B} + \mathcal{B}^*\mathcal{B}')\,\|\mathcal{B}\|^{-4}\mathcal{B}^* \,+\, \mathcal{A}\,\|\mathcal{B}\|^{-2}\mathcal{B}'^*\,.$$

Multiplying both sides by $\|\mathcal{B}\|^4$ we obtain

$$\left(\mathcal{A}\mathcal{B}^{-1}\right)'\|\mathcal{B}\|^4 = \mathcal{A}'\|\mathcal{B}\|^2\mathcal{B}^* - \mathcal{A}\mathcal{B}'^*\|\mathcal{B}\|^2 - \mathcal{A}\mathcal{B}^*\mathcal{B}'\mathcal{B}^* + \mathcal{A}\|\mathcal{B}\|^2\mathcal{B}'^* = (\mathcal{A}'\|\mathcal{B}\|^2 - \mathcal{A}\mathcal{B}^*\mathcal{B}')\mathcal{B}^*\,.$$

On the other hand (9) implies that $\mathcal{A}^* \left(\|\mathcal{B}\|^2 \mathcal{A}' - \mathcal{A}\mathcal{B}^*\mathcal{B}' \right) = 0$, and since $\mathcal{A} \neq 0$ we obtain $(\mathcal{A}\mathcal{B}^{-1})' = 0$, which in turn implies that $\mathcal{A}\mathcal{B}^{-1} = \mathcal{C}$, or $\mathcal{A} = \mathcal{C}\,\mathcal{B}$, as required. ■

We elaborate below on the form of condition (9) for specific instances of F.

If $F = \mathbb{R}$ and $\mathcal{A}(t) = a(t)$, $\mathcal{B}(t) = b(t)$, $\mathcal{C} = \gamma$ satisfy (8), condition (9) is just $b^2(t)a(t)a'(t) = a^2(t)b(t)b'(t)$, which is equivalent to the Wronskian $W(a, b)$ vanishing. The situation is more interesting when $F = \mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. In fact, if we consider only *primitive* elements of $F[t]$ — i.e., those for which the greatest common divisor of their components is equal to 1 — the imaginary part of condition (9) suffices.

Theorem 3.1 *Let $\mathcal{A}(t)$, $\mathcal{B}(t)$ be non–zero elements of $F[t]$, where $F = \mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Then $\mathcal{A}(t) = r(t)\,\mathcal{C}\,\mathcal{B}(t)$, where $r(t)$ is a real rational function and $\mathcal{C} \in F$, if and only if*

$$\|\mathcal{B}\|^2\, \mathrm{Im}(\mathcal{A}^*\mathcal{A}') \;=\; \|\mathcal{A}\|^2\, \mathrm{Im}(\mathcal{B}^*\mathcal{B}')\,. \tag{10}$$

In particular, if $\mathcal{A}(t)$, $\mathcal{B}(t)$ are primitive*, then $r(t) = 1$.*

The above is motivated by the case $F = \mathbb{C}$, as the following remark shows.

Remark 3.1 *Let $\mathcal{A}(t) = a(t) + \mathrm{i}\, b(t)$ and $\mathcal{B}(t) = c(t) + \mathrm{i}\, d(t)$ be primitive elements of $\mathbb{C}[t]$. Then, $\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$ if and only if*

$$\|\mathcal{B}\|^2\, \mathrm{Im}(\mathcal{A}^*\mathcal{A}') \;=\; \|\mathcal{A}\|^2\, \mathrm{Im}(\mathcal{B}^*\mathcal{B}')\,, \qquad \text{i.e.,} \quad \frac{ab' - a'b}{a^2 + b^2} = \frac{cd' - c'd}{c^2 + d^2}\,. \tag{11}$$

To see this, observe that (11) is equivalent to $uv' - u'v = 0$, where $u = ac + bd$, $v = bc - ad$. Thus, there exists $\lambda \in \mathbb{R}$ such that $u = \lambda\, v$, i.e.,

$$a(c + \lambda d) \;=\; b(\lambda c - d) \qquad \text{or} \qquad c(a - \lambda b) \;=\; -\,d(\lambda a + b)\,. \tag{12}$$

Since $\gcd(a, b) = \gcd(c, d) = 1$, we may infer that $a \,|\, \lambda c - d$, $b \,|\, c + \lambda d$, $c \,|\, \lambda a + b$, $d \,|\, a - \lambda b$. Assuming, without loss of generality, that $\deg(b) \le \deg(a)$ and $\deg(d) \le \deg(c)$, from the above we also have $\deg(c) \le \deg(a)$ and $\deg(d) \le \deg(a)$, and thus $\deg(\lambda c - d) \le \deg(a)$. Since $\deg(a) \le \deg(\lambda c - d)$, we must have $\deg(a) = \deg(\lambda c - d)$. Analogous arguments show that $\deg(b) = \deg(c + \lambda d)$. In view of the above and (12), $a = \tau(\lambda c - d)$ and $b = \tau(c + \lambda d)$ for some $\tau \in \mathbb{R}$. Thus, $a + \mathrm{i}\, b = (\lambda\tau + \mathrm{i}\,\tau)(c + \mathrm{i}\, d)$, as required. Finally, note that Proposition 4.2 in [11] provides a much simpler (but more *restrictive*) proof of this remark. ◊

Proof of Theorem 3.1: We shall prove the theorem when $F = \mathbb{H}$ (the case $F = \mathbb{O}$ is argued similarly). Let $\mathcal{A}(t) = a(t) + \mathbf{i}\,b(t) + \mathbf{j}\,c(t) + \mathbf{k}\,d(t)$, $\mathcal{B}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t) \in \mathbb{H}[t]$.

- First, consider the case where $\mathcal{A}(t)$, $\mathcal{B}(t)$ are primitive. Condition (10) is equivalent to

$$\begin{aligned} \|\mathcal{B}\|^2\,(a'b - ab' - c'd + cd') &= \|\mathcal{A}\|^2\,(u'v - uv' - p'q + pq')\,, \\ \|\mathcal{B}\|^2\,(a'c - ac' + b'd - bd') &= \|\mathcal{A}\|^2\,(u'p - up' + v'q - vq')\,, \\ \|\mathcal{B}\|^2\,(a'd - ad' - b'c + bc') &= \|\mathcal{A}\|^2\,(u'q - uq' - v'p + vp')\,. \end{aligned} \tag{13}$$

If $\mathcal{A}$, $\mathcal{B}$ satisfy (13), set $\mathcal{A}'^*\mathcal{A} = a_0 + \mathbf{i}\,a_1 + \mathbf{j}\,a_2 + \mathbf{k}\,a_3$, $\mathcal{B}'^*\mathcal{B} = b_0 + \mathbf{i}\,b_1 + \mathbf{j}\,b_2 + \mathbf{k}\,b_3$, $f = \|\mathcal{A}\|^2$, $g = \|\mathcal{B}\|^2$, $r = a'^2 + b'^2 + c'^2 + d'^2$ and $s = u'^2 + v'^2 + p'^2 + q'^2$.

Let ρ be a root of g with multiplicity k. We wish to show that ρ is also a root of f, with the same multiplicity. Suppose first that ρ is not a root of f. Then b_1, b_2, b_3 must vanish at $t = \rho$. In addition, since

$$b_0^2 + b_1^2 + b_2^2 + b_3^2 \,=\, gs \qquad \text{or} \qquad \frac{b_0^2 + b_1^2 + b_2^2 + b_3^2}{g^2} \,=\, \frac{s}{g}\,, \tag{14}$$

we see that ρ must also be a root of b_0. Now $\|\mathcal{B}\|^2\,\mathrm{Im}(\mathcal{A}^*\mathcal{A}') = \|\mathcal{A}\|^2\,\mathrm{Im}(\mathcal{B}^*\mathcal{B}')$ implies that

$$\frac{b_1}{g} = \frac{a_1}{f}\,, \qquad \frac{b_2}{g} = \frac{a_2}{f}\,, \qquad \frac{b_3}{g} = \frac{a_3}{f}\,. \tag{15}$$

Since ρ is not a root of f, the above imply that it cannot be a pole of the rational functions b_1/g, b_2/g, b_3/g, and hence ρ must be a root of b_1, b_2, b_3 of multiplicity k at least. Since $\mathcal{B}$ is primitive, assume that $u(\rho) \neq 0$. Now b_1, b_2, b_3 are the expressions multiplying $\|\mathcal{A}\|^2$ on the right in (13), and one can easily verify that they satisfy $u'b_0 - v'b_1 - p'b_2 - q'b_3 = u\,s$. Also, ρ must be a root of b_0 with multiplicity $k-1$, since $b_0 = \frac{1}{2}\,g'$. Hence, the multiplicity of ρ as a root of s is at least $k-1$. But this contradicts (14) since ρ is a pole (of order two) of b_0^2/g^2. Therefore, ρ is a root of f as well.

Now let the multiplicity of a root δ of a polynomial $\phi(t) \in \mathbb{C}[t]$ be denoted by $m(\delta, \phi)$, and assume that $k = m(\rho, g) > m(\rho, f) = \kappa$, and thus $k \geq 2$. Since $\mathcal{B}$ is primitive, we may assume, through multiplication by a suitable $\mathcal{U} \in \mathbb{H}$, that

$$uvpq\,(u^2 + v^2)\,(u^2 + p^2)\,(u^2 + q^2)\,\Big|_{t=\rho} \,\neq\, 0\,.$$

Observe that $ub_1 + qb_2 - pb_3 = -\,v\,b_0 + v'g$ and $b_1 = b_2 = b_3 = 0$ at $t = \rho$. Since b_0/g has ρ as a pole of order 1, we deduce that at least one of b_1/g, b_2/g, b_3/g must have ρ as a pole of order 1 as well. Assume, for the moment, that $m(\rho, b_1) = k-1$. In addition, we have

$$qb_2 - pb_3 \;=\; -ub_1 - vb_0 + v'g\,, \qquad pb_2 + qb_3 \;=\; vb_1 - ub_0 + u'g\,.$$

Since $(p^2+q^2)\,|_{t=\rho} \neq 0$, we see that $m(\rho, b_2)$ and $m(\rho, b_3)$ are at least $k-1$. Recall that $v'b_1 - p'b_2 - q'b_3 + u'b_0 = u\,s$, and thus $m(\rho, s) \geq k-1$. But this again contradicts (14).

The cases $m(\rho, b_i) = k-1$ for $i = 2, 3$ are treated similarly. Thus, since every root of $\|\mathcal{B}(t)\|^2$ is also a root of $\|\mathcal{A}(t)\|^2$ with the same multiplicity, $\|\mathcal{A}(t)\|^2$ is a constant multiple of $\|\mathcal{B}(t)\|^2$, and therefore $\|\mathcal{B}\|^2\,\mathrm{Re}(\mathcal{A}^*\mathcal{A}') = \|\mathcal{A}\|^2\,\mathrm{Re}(\mathcal{B}^*\mathcal{B}')$. This together with (10) implies (9), as required.

• Suppose now that $\mathcal{A}(t)$, $\mathcal{B}(t)$ are non–primitive. Then we set $\mathcal{A} = \alpha\,\mathcal{A}_1$ and $\mathcal{B} = \beta\,\mathcal{B}_1$ where $\alpha = \gcd(a,b,c,d)$ and $\beta = \gcd(u,v,p,q)$. Here $\mathcal{A}_1$, $\mathcal{B}_1$ are primitive, and a simple calculation shows that they satisfy (13). Hence, $\mathcal{A}_1 = \mathcal{C}\,\mathcal{B}_1$ for some $\mathcal{C} \in \mathbb{H}$. This implies that $\mathcal{A} = r\,\mathcal{C}\,\mathcal{B}$, where $r = \alpha/\beta$, as required. Conversely, suppose $\mathcal{A}(t) = r(t)\,\mathcal{C}\,\mathcal{B}(t)$. Then $\|\mathcal{B}\|^2\,\mathcal{A}^*\mathcal{A}' = rr'\,\|\mathcal{C}\|^2\,\|\mathcal{B}\|^2 + \|\mathcal{A}\|^2\,\mathcal{B}^*\,\mathcal{B}'$, and thus $\|\mathcal{B}\|^2\,\mathrm{Im}(\mathcal{A}^*\mathcal{A}') = \|\mathcal{A}\|^2\,\mathrm{Im}(\mathcal{B}^*\mathcal{B}')$. ∎

Results analogous to those of Proposition 3.1 and Theorem 3.1 can be obtained in the case where $\mathcal{A}(t) = \mathcal{B}(t)\,\mathcal{C}$.

Note: Evidently, there are various equivalent forms of condition (9). We opt for the specific form in (9) for two reasons: (a) it is symmetric with respect to $\mathcal{A}$ and $\mathcal{B}$; and (b) for primitive $\mathcal{A}(t) \in \mathbb{H}[t]$ and $\mathcal{B}(t) \in \mathbb{C}[t]$, equality between the $\mathbf{i}$ component of $\|\mathcal{A}\|^{-2}\,\mathcal{A}^*\mathcal{A}'$ and $\|\mathcal{B}\|^{-2}\,\mathrm{Im}(\mathcal{B}^*\mathcal{B}')$ is Han's condition for the PH curve defined by $\mathcal{A}$ to have a rational rotation–minimizing frame — see, for example, formula (8) on page 847 of [11].

3.2 An application

• Linear dependence of two vectors. Let $\mathsf{a} = (a_0, \ldots, a_m), \mathsf{b} = (b_0, \ldots, b_m) \in \mathbb{R}^{m+1}$. It is well–known that a and b are linearly dependent if and only if the determinants of all 2×2 minors of the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \end{bmatrix}$$

vanish. This gives $\frac{1}{2}m(m+1)$ conditions, that are quadratic in $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$. We can, however, substantially reduce this number as follows. Set $f_\mathsf{a}(t) = a_0 + a_1 t + \cdots + a_m t^m$, $f_\mathsf{b}(t) = b_0 + b_1 t + \cdots + b_m t^m$. Then a, b are linearly dependent if and only if $f_\mathsf{a}(t)$, $f_\mathsf{b}(t)$ are

— i.e., if $f_{\mathsf{ab}}(t) := f'_{\mathsf{a}}(t)\, f_{\mathsf{b}}(t) - f_{\mathsf{a}}(t)\, f'_{\mathsf{b}}(t) \equiv 0$. Since $\deg(f_{\mathsf{ab}}) = 2m-2$, we obtain $2m-1$ conditions, significantly less than $\frac{1}{2}m(m+1)$.

Note that when $m = 2$, the conditions obtained are: $(a_1b_0 - a_0b_1, 2(a_2b_0 - b_2a_0), a_2b_1 - a_1b_2) = (0,0,0)$, and this is equivalent to the cross product $\mathsf{a} \times \mathsf{b} = \vec{0}$. Also, for $m = 3$, the above procedure gives 5 conditions, while the cross product $\mathsf{a} \times \mathsf{b}$ gives 6. On the other hand, if $4 \le m \le 6$, we get precisely 7 conditions from the cross product $\mathsf{a} \times \mathsf{b} = \vec{0}$, much less than $\frac{1}{2}4(4+1) = 10$, $\frac{1}{2}5(5+1) = 15$ or $\frac{1}{2}6(6+1) = 21$ from the above procedure.

These considerations lead us to pose the following problem.

Question 1 *Given* $\mathsf{a} = (a_0, \ldots, a_m), \mathsf{b} = (b_0, \ldots, b_m) \in \mathbb{R}^{m+1}$ *what is the* minimum *number of conditions, quadratic in* $a_0, \ldots, a_m$ *and* $b_0, \ldots, b_m$*, such that* a*,* b *are linearly dependent?*

• Linear dependence in $\mathbb{R}[t]$. When $F = \mathbb{R}$, the Wronskian can be used to develop conditions that identify when three or more elements of $\mathbb{R}[t]$ are linearly dependent [1]. Here, we state a result concerning three–element dependency.

Proposition 3.2 *Three polynomials* $a(t)$*,* $b(t)$*,* $c(t) \in \mathbb{R}[t]$ *are linearly dependent if and only if they satisfy*

$$(ac' - a'c)(bc'' - b''c) \;=\; (ac'' - a''c)(bc' - b'c)\,. \tag{16}$$

Proof: For polynomials $a(t)$, $b(t)$, $c(t)$ the Wronskian can be written as

$$W(a,b,c) = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = [\,(ac' - a'c)(bc'' - b''c) - (ac'' - a''c)(bc' - b'c)\,]\,c\,.$$

Thus, when $c(t) \not\equiv 0$, the polynomials $a(t)$, $b(t)$, $c(t)$ are linearly dependent if and only if condition (16) is satisfied. ■

4 Linear and planar PH curves

As in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$, it is useful to know when a higher–dimension PH curve $\mathbf{r}(t)$ is linear or planar. We now apply the results of the preceding section to address this problem. Let $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t)) \,:\, \mathbb{R} \to \mathbb{R}^n$ be a PH curve of dimension n. Then there exist $\mathcal{A}(t), \mathcal{B}(t) \in R_{n-1}[t]$ that satisfy (5). We first deal with the linear case.

• Linear case. If all components of $\mathbf{r}'(t)$ vanish, $\mathbf{r}(t)$ is just a single point. Assume then that some component of $\mathbf{r}'(t)$, denoted by $\delta(t)$, is non–zero. Then $\mathbf{r}(t)$ is a straight line if and only if all components of $\mathbf{r}'(t)$ are scalar multiples of $\delta(t)$. Suppose first that $x_1'(t) = \|\mathcal{A}(t)\|^2 - \|\mathcal{B}(t)\|^2 \neq 0$, and take this to be $\delta(t)$. Then $\mathcal{A}(t)\,\mathcal{B}^*(t) = \delta(t)\,\mathbf{v}$ where $\mathbf{v} \in \mathbb{R}^{n-1}$, which implies that $\delta^2(t) = \nu\,\|\mathcal{A}(t)\|^2\,\|\mathcal{B}(t)\|^2$ for some $\nu > 0$. But this is possible if and only if $\|\mathcal{A}(t)\| = c\,\|\mathcal{B}(t)\|$ for $c \in \mathbb{R}$. Therefore, $\delta(t) = \gamma\,\|\mathcal{B}(t)\|^2$, and from $\mathcal{A}(t)\,\mathcal{B}^*(t) = \delta(t)\,\mathbf{v}$ we then have

$$\mathcal{A}(t) \;=\; \frac{\delta(t)\,\mathbf{v}\,\mathcal{B}(t)}{\|\mathcal{B}(t)\|^2} \;=\; \mathcal{C}\,\mathcal{B}(t)\,, \quad \text{where } \mathcal{C} \in R_n\,. \tag{17}$$

Now suppose that $\|\mathcal{A}(t)\|^2 - \|\mathcal{B}(t)\|^2 = 0$. Then, $\delta(t)$ is one of the components of $\mathcal{A}(t)\,\mathcal{B}^*(t)$. Again, we have $\mathcal{A}(t)\,\mathcal{B}^*(t) = \delta(t)\,\mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^{n-1}$, and thus $\delta(t) = \mu\,\|\mathcal{B}(t)\|^2$, which brings us to the previous case.

Notice that since $\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$ and in (5) we may take $\gcd(\mathcal{A},\mathcal{B}) = 1$, we see that $\mathcal{A}$ and $\mathcal{B}$ are both primitive. We summarize the above as follows.

Proposition 4.1 *Let $\mathbf{r}(t)$ be a PH curve with hodograph $\mathbf{r}'(t)$ defined by* (5)*, where we have* $\gcd(\mathcal{A}(t),\mathcal{B}(t)) = 1$*. Then $\mathbf{r}(t)$ is linear if and only if $\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$, where $\mathcal{C} \in R_n$ and $\mathcal{A}(t)$, $\mathcal{B}(t)$ are both* primitive*.*

The linearity condition $\mathcal{A}(t) = \mathcal{C}\,\mathcal{B}(t)$ involves the unknown coefficient $\mathcal{C}$. As observed in Section 3, we may eliminate it using Proposition 3.1 to obtain the equivalent condition

$$\|\mathcal{B}\|^2\,\mathcal{A}^*\mathcal{A}' \;=\; \|\mathcal{A}\|^2\,\mathcal{B}^*\mathcal{B}'\,. \tag{18}$$

Conditions (17) and (18) become much simpler if we map $\mathbf{r}(t)$ to *normal form*. Let $\mathbf{r}'(t) = H(\mathcal{A}(t),\mathcal{B}(t))$ and write $\mathcal{A}(t) = \mathcal{A}_0 + \mathcal{A}_1 t + \cdots + \mathcal{A}_m t^m$ and $\mathcal{B}(t) = \mathcal{B}_0 + \mathcal{B}_1 t + \cdots + \mathcal{B}_m t^m$. If we replace $\mathcal{A}(t)$, $\mathcal{B}(t)$ with $\mathcal{M}\mathcal{A}(t) - \mathcal{N}^*\mathcal{B}(t)$, $\mathcal{N}\mathcal{A}(t) + \mathcal{M}^*\mathcal{B}(t)$, where

$$\mathcal{M} \;=\; \frac{\mathcal{A}_m^*}{\|\mathcal{A}_m\|^2 + \|\mathcal{B}_m\|^2} \qquad \text{and} \qquad \mathcal{N} \;=\; \frac{-\,\mathcal{B}_m}{\|\mathcal{A}_m\|^2 + \|\mathcal{B}_m\|^2}\,,$$

then $\mathcal{A}(t)$ becomes monic of degree m, while $\mathcal{B}(t)$ is of degree less than m. Henceforth, we call such a pair $(\mathcal{A}(t),\mathcal{B}(t))$ *normal*.

In view of the above, we immediately have the following.

Corollary 4.1 *Let $\mathbf{r}(t)$ be a PH curve with hodograph defined by* (5)*, where* $(\mathcal{A}(t),\mathcal{B}(t))$ *are* normal*. Then, $\mathbf{r}(t)$ is linear if and only if $\mathcal{B}(t) = 0$.*

Proof: If $\mathcal{B}(t) \neq 0$, any non–zero component $\delta(t)$ of $\mathcal{A}(t)\,\mathcal{B}^*(t)$ is linearly independent of $\|\mathcal{A}(t)\|^2 - \|\mathcal{B}(t)\|^2$, since $\deg(\delta) < \deg(\|\mathcal{A}\|^2 - \|\mathcal{B}\|^2) = 2m$. ■

• Planar case. Assume that $\mathbf{r}(t)$ is in normal form. Then it is (truly) planar if and only if the components of $\mathcal{A}(t)\,\mathcal{B}^*(t)$ define a (non–degenerate) line. Thus, there exists a non–zero component $\delta(t)$ of $\mathcal{A}(t)\,\mathcal{B}^*(t)$ such that $\mathcal{A}(t)\,\mathcal{B}^*(t) = \delta(t)\,\mathbf{v}$ for $\mathbf{v} \in \mathbb{R}^{n-1}$. This implies that $\mathcal{A}(t) = \delta(t)\,\mathbf{v}\,\mathcal{B}(t)/\|\mathcal{B}(t)\|^2$. As before, this condition is coefficient–dependent. However, we can use Theorem 3.1 to obtain a condition involving only $\mathcal{A}(t)$ and $\mathcal{B}(t)$.

Proposition 4.2 *Let $\mathbf{r}(t)$ be a PH curve with hodograph defined by* (5)*, where $(\mathcal{A}(t), \mathcal{B}(t))$ are* normal*. Then $\mathbf{r}(t)$ is (truly) planar if and only if*

$$\mathcal{A}(t) \,=\, r(t)\,\mathcal{C}\,\mathcal{B}(t) \quad \text{or} \quad \|\mathcal{B}\|^2\,\mathrm{Im}(\mathcal{A}^*\mathcal{A}') \,=\, \|\mathcal{A}\|^2\,\mathrm{Im}(\mathcal{B}^*\mathcal{B}')\,, \tag{19}$$

where $\mathcal{C} \in R_{n-1}$.

5 Closure

A Pythagorean–hodograph (PH) curve in $\mathbb{R}^n$ is a polynomial curve $\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^n$ whose derivative $\mathbf{r}'(t)$ has components $x_1'(t), \ldots, x_n'(t)$ that satisfy the Pythagorean condition (1). Motivated by the computational advantages that this structure confers in computer–aided design, computer graphics, animation, motion control, robotics, and similar applications, PH curves have thus far been extensively studied in $\mathbb{R}^2$ and $\mathbb{R}^3$. The present paper describes a unifying framework for constructing PH curves in higher dimensions, using a generalized Hopf map model. In particular, the PH curves in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^5$ and $\mathbb{R}^9$ are generated in a natural manner through the real, complex, quaternion, and octonion algebras, and some other dimensions can be accommodated by specialization of these models. These models also furnish simple criteria that allow one to identify when higher–dimension PH curves degenerate to lower–dimension (linear or planar) loci.

Many open problems remain concerning these higher–dimensional PH curves: for example, the identification of conditions under which they degenerate to loci in $\mathbb{R}^3$ or other subspaces of $\mathbb{R}^n$, and the characterization of subsets of these curves analogous to the helical polynomial curves [7, 13] and rational rotation–minimizing frame curves [5, 11] in $\mathbb{R}^3$.

Appendix: Quaternions and Octonions

A brief review of the quaternion and octonion algebras is provided here, for readers who are not already familiar with them. Quaternions are "four–dimensional numbers" of the form $\mathcal{A} = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where a is the real (scalar) and $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is the imaginary (vector) part. The conjugate of $\mathcal{A}$ is $\mathcal{A}^* = a - a_x\mathbf{i} - a_y\mathbf{j} - a_z\mathbf{k}$, and its norm is the non–negative value $\|\mathcal{A}\|$ specified by $\|\mathcal{A}\|^2 = \mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$. The basis elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and

$$\mathbf{i}\,\mathbf{j} = -\,\mathbf{j}\,\mathbf{i} = \mathbf{k}\,, \qquad \mathbf{j}\,\mathbf{k} = -\,\mathbf{k}\,\mathbf{j} = \mathbf{i}\,, \qquad \mathbf{k}\,\mathbf{i} = -\,\mathbf{i}\,\mathbf{k} = \mathbf{j}\,.$$

Consequently, the quaternion product is not commutative, $\mathcal{A}\,\mathcal{B} \neq \mathcal{B}\mathcal{A}$ in general, but it is associative, i.e., $(\mathcal{A}\,\mathcal{B})\,\mathcal{C} = \mathcal{A}\,(\mathcal{B}\,\mathcal{C})$ for any $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

Octonions are "eight–dimensional numbers," $\mathcal{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4 + a_5\mathbf{e}_5 + a_6\mathbf{e}_6 + a_7\mathbf{e}_7$, where a_0 is the real part and $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4 + a_5\mathbf{e}_5 + a_6\mathbf{e}_6 + a_7\mathbf{e}_7$ is the imaginary part. $\mathcal{A}$ has the conjugate $\mathcal{A}^* = a_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3 - a_4\mathbf{e}_4 - a_5\mathbf{e}_5 - a_6\mathbf{e}_6 - a_7\mathbf{e}_7$ and the non–negative norm specified by $\|\mathcal{A}\|^2 = \mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$. The basis elements $\mathbf{e}_1, \ldots, \mathbf{e}_7$ satisfy $\mathbf{e}_1^2 = \cdots = \mathbf{e}_7^2 = -1$ and

$$\mathbf{e}_k\mathbf{e}_{k+1} = -\,\mathbf{e}_{k+1}\mathbf{e}_k = \mathbf{e}_{k+3}, \quad \mathbf{e}_{k+1}\mathbf{e}_{k+3} = -\,\mathbf{e}_{k+3}\mathbf{e}_{k+1} = \mathbf{e}_k, \quad \mathbf{e}_{k+3}\mathbf{e}_k = -\,\mathbf{e}_k\mathbf{e}_{k+3} = \mathbf{e}_{k+1}$$

for $k = 1, \ldots, 7$ (where all indices are reduced modulo 7). The octonion product is neither commutative nor associative — in general, $\mathcal{A}\,\mathcal{B} \neq \mathcal{B}\mathcal{A}$ and $(\mathcal{A}\,\mathcal{B})\,\mathcal{C} \neq \mathcal{A}\,(\mathcal{B}\,\mathcal{C})$.

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