Rational rotation-minimizing frames on polynomial space curves of arbitrary degree

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Abstract

A rotation–minimizing adapted frame on a space curve $\mathbf{r}(t)$ is an orthonormal basis $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ for $\mathbb{R}^3$, such that $\mathbf{f}_1$ is coincident with the curve tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ at each point, and the normal–plane vectors $\mathbf{f}_2$, $\mathbf{f}_3$ exhibit no instantaneous rotation about $\mathbf{f}_1$. Such frames are of interest in applications such as spatial path planning, computer animation, robotics, and swept surface constructions. Polynomial curves with *rational* rotation–minimizing frames (RRMF curves) are necessarily Pythagorean–hodograph (PH) curves — since only the PH curves possess rational unit tangents — and may be characterized by constraints on the coefficients of the quaternion or Hopf map forms of spatial PH curves. As a generalization of prior characterizations for RRMF cubics and quintics, a sufficient–and–necessary condition for spatial PH curves of arbitrary degree to be RRMF curves is derived herein. This RRMF condition amounts to a divisibility property for certain combinations of polynomials formed from the components of the quaternion or Hopf map representations of spatial PH curves.

Keywords: rotation–minimizing frames; Pythagorean–hodograph curves; spatial motion planning; quaternions; Hopf map; polynomial identities.

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1 Introduction

An *adapted frame* on a regular space curve $\mathbf{r}(t)$ is an orthonormal basis $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ for $\mathbb{R}^3$, such that $\mathbf{f}_1$ coincides with the curve tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ at each point. The *Frenet frame* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is perhaps the most familiar adapted frame [18], for which the principal normal $\mathbf{n}$ points toward the *center of curvature*, and the binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is orthogonal to the *osculating plane*. However, the Frenet frame, defined in terms of the *intrinsic geometry* at each curve point, is often unsuitable for specifying the orientation of a rigid body along a given curve in applications such as motion planning, animation, geometric design, and robotics, since it incurs "unnecessary" rotation of the body.

As noted by Bishop [1], there are infinitely many adapted frames on a given space curve, since they possess a residual freedom, controlling the orientation of the frame vectors $\mathbf{f}_2, \mathbf{f}_3$ in the curve normal plane. The variation of an adapted frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ along a curve $\mathbf{r}(t)$ is determined by its vector angular velocity $\boldsymbol{\omega}(t)$ through the differential relations

$$\frac{\mathrm{d}\mathbf{f}_1}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{f}_1 \,, \qquad \frac{\mathrm{d}\mathbf{f}_2}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{f}_2 \,, \qquad \frac{\mathrm{d}\mathbf{f}_3}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{f}_3 \,,$$

where s is arc length along the curve. The magnitude $\omega = |\boldsymbol{\omega}|$ and direction $\mathbf{a} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ of the angular velocity specify the instantaneous angular speed and rotation axis of the frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. The characteristic property of a *rotation–minimizing* adapted frame is that its angular velocity has no component along $\mathbf{f}_1 = \mathbf{t}$ — i.e., $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$ (there is no instantaneous rotation of $\mathbf{f}_2$ and $\mathbf{f}_3$ about $\mathbf{f}_1 = \mathbf{t}$). The Frenet frame is not rotation–minimizing, since its angular velocity is given [18] by the *Darboux vector* $\boldsymbol{\omega} = \kappa\,\mathbf{b} + \tau\,\mathbf{t}$, where κ and τ are the curvature and torsion of $\mathbf{r}(t)$.

Klok [17] studied rotation–minimizing frames (RMFs) in the construction of swept surfaces, and characterized them as solutions of first–order differential equations. Guggenheimer [12] subsequently showed that the RMF normal–plane vectors have an orientation, relative to the Frenet frame, defined (modulo a constant) by the integral of the torsion with respect to arc length, which must generally be computed by numerical quadrature. Further details on RMF approximations and applications may be found in [11, 14, 15, 16, 21, 22, 23].

Pythagorean–hodograph (PH) *curves* [6] permit exact RMF computation [5] by integrating a rational function, but this typically incurs transcendental terms. More recently, there has been interest in constructing polynomial curves with *rational rotation–minimizing frames* (RRMF curves). Such curves must be PH curves, since only the PH curves have rational unit tangents — the RRMF curves should be identifiable by constraints on the coefficients of PH curves that are sufficient and necessary for the RMF to be rational. Rational forms are always preferable, whenever possible, since they admit exact and efficient computation.

As an alternative to invoking the Frenet frame as a reference, Choi and Han [2] defined an adapted frame on spatial PH curves called the *Euler–Rodrigues frame* (ERF). This is not a geometrically intrinsic frame (it depends on the chosen coordinate system), but it offers two key advantages over the Frenet frame for identifying rational RMFs — it is inherently rational, and always non–singular at inflection points. The conditions under which ERFs can be RMFs were studied in [2], showing that: (a) for PH cubics, the ERF and the Frenet frame are equivalent; (b) for PH quintics, the ERF is an RMF only for degenerate (planar) curves; and (c) PH curves for which the ERF is an RMF are of degree 7 at least.

Subsequently, Han [13] used the ERF to identify a criterion characterizing RRMF curves of any (odd) degree, and showed that RRMF cubics are degenerate PH cubics, i.e., they are either planar, or have non–primitive hodographs. The existence of non–degenerate RRMF quintics was first shown in [8], using the Hopf map representation of spatial PH curves rather than the more–familiar quaternion form. Much simpler characterizations of RRMF quintics were then derived in [7], in terms of both the quaternion and Hopf map forms, that are just quadratic in the curve coefficients and incorporate expected symmetry properties.

The goal of the present paper is to determine general constraints on the four polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ specifying the quaternion and Hopf map forms of spatial PH curves, that are sufficient and necessary for the existence of a rational RMF, and incorporate the known results concerning RRMF cubics and quintics as special instances.

The paper is organized as follows. Section 2 reviews the quaternion and Hopf map forms of spatial PH curves, and the general RRMF criterion expressed in terms of them. Conditions under which spatial PH curves degenerate to straight lines or planar curves, which are trivial RRMF curves, are then determined in Section 3. The main results concerning derivation of sufficient and necessary conditions for non–degenerate RRMF curves, expressed in terms of the divisibility of certain polynomials, are then presented in Section 4, and prior results concerning RRMF cubics and quintics are newly interpreted in terms of this condition. Finally, Section 5 makes some preliminary remarks concerning higher–order RRMF curves, while Section 6 summarizes the results of this paper and identifies topics for further study.

2 Quaternion and Hopf map forms

A polynomial Pythagorean–hodograph curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ is characterized by the property that its derivative components satisfy the Pythagorean condition

$$x'^2(t) + y'^2(t) + z'^2(t) \;=\; \sigma^2(t) \tag{1}$$

for some polynomial $\sigma(t)$. Hence, the parametric speed $|\mathbf{r}'(t)|$, which defines the rate of change $\mathrm{d}s/\mathrm{d}t$ of arc length s with respect to the parameter t, is defined by the polynomial $\sigma(t)$ — rather than the square–root of a polynomial. A sufficient–and-necessary condition for the satisfaction of (1) is that $x'(t)$, $y'(t)$, $z'(t)$ should be expressible [3, 4] in terms of four polynomials[1] $u(t)$, $v(t)$, $p(t)$, $q(t)$ in the form

$$\begin{aligned} x'(t) \;&=\; u^2(t) + v^2(t) - p^2(t) - q^2(t)\,, \\ y'(t) \;&=\; 2\,[\,u(t)q(t) + v(t)p(t)\,]\,, \\ z'(t) \;&=\; 2\,[\,v(t)q(t) - u(t)p(t)\,]\,, \end{aligned} \tag{2}$$

and the parametric speed is then given by

$$\sigma(t) \;=\; u^2(t) + v^2(t) + p^2(t) + q^2(t)\,. \tag{3}$$

The Pythagorean hodograph structure (2) in $\mathbb{R}^3$ is conveniently captured by two algebraic models, introduced by Choi et al. [3]. A Pythagorean hodograph may be generated from a quaternion[2] polynomial

$$\mathcal{A}(t) \;=\; u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k} \tag{4}$$

by the product

$$\mathbf{r}'(t) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)\,, \tag{5}$$

where $\mathcal{A}^*(t) = u(t) - v(t)\,\mathbf{i} - p(t)\,\mathbf{j} - q(t)\,\mathbf{k}$ is the quaternion conjugate of $\mathcal{A}(t)$. This may be regarded as generating $\mathbf{r}'(t)$, for each t, by a scaling/rotation operation applied to the unit vector $\mathbf{i}$. Note that the expression on the right in (5) is a quaternion with zero real (scalar) part, regarded as a vector in $\mathbb{R}^3$.

[1] We assume, throughout the paper, that these polynomials satisfy $\gcd(u(t), v(t), p(t), q(t)) = 1$.

[2] We use calligraphic characters for quaternions, and bold font for vectors in $\mathbb{R}^3$ and occasionally also for complex numbers — the meaning should be clear from the context. The scalar and vector parts [20] of a quaternion $\mathcal{A}$ are denoted by $\mathrm{scal}(\mathcal{A})$ and $\mathrm{vect}(\mathcal{A})$.

Alternatively, Pythagorean hodographs may be generated from complex polynomial pairs

$$\boldsymbol{\alpha}(t) \,=\, u(t) + \mathrm{i}\, v(t)\,, \quad \boldsymbol{\beta}(t) \,=\, q(t) + \mathrm{i}\, p(t) \tag{6}$$

by the expression

$$\mathbf{r}'(t) \,=\, (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)))\,. \tag{7}$$

This corresponds to the *Hopf map* from $\mathbb{C}^2$ (or $\mathbb{R}^4$) to $\mathbb{R}^3$. The equivalence of (5) and (7) is established by taking $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$, where the imaginary unit i is identified with the quaternion element $\mathbf{i}$. The forms (5) and (7) are both *invariant* with respect arbitrary spatial rotations — i.e., a given Pythagorean hodograph can always be expressed in both these forms, regardless of the orientation of the adopted coordinates in $\mathbb{R}^3$.

As previously noted, curves with rational rotation–minimizing frames (RRMF curves) are necessarily PH curves. Han [13] derived a sufficient and necessary condition for a PH curve to possess a rational RMF in terms of the quaternion form — the hodograph (5) defines an RRMF curve if and only if two relatively prime polynomials $a(t)$, $b(t)$ exist, such that the components $u(t)$, $v(t)$, $p(t)$, $q(t)$ of $\mathcal{A}(t)$ satisfy

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} \,=\, \frac{ab' - a'b}{a^2 + b^2}\,. \tag{8}$$

Note that the numerator and denominator of the expression on the left can be written in terms of $\mathcal{A}(t)$ as $\mathrm{scal}(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t))$ and $|\mathcal{A}(t)|^2$.

For the Hopf map form, the condition (8) can be regarded [8] as requiring the existence of a complex polynomial $\mathbf{w}(t) = a(t) + \mathrm{i}\, b(t)$ with $\gcd(a(t), b(t)) = 1$, such that

$$\frac{\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' - \overline{\boldsymbol{\alpha}}'\boldsymbol{\alpha} + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}' - \overline{\boldsymbol{\beta}}'\boldsymbol{\beta}}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} \,=\, \frac{\overline{\mathbf{w}}\mathbf{w}' - \overline{\mathbf{w}}'\mathbf{w}}{|\mathbf{w}|^2}\,. \tag{9}$$

In this paper, we shall consider satisfaction of (8) in the cases where (a) $\gcd(uv' - u'v - pq' + p'q, u^2 + v^2 + p^2 + q^2) = 1$, and (b) $u^2 + v^2 + p^2 + q^2 = a^2 + b^2$.

3 Degenerate RRMF curves

Straight lines are trivially RRMF curves, since we need only choose a unit vector $\mathbf{f}_1$ along the line, and two unit vectors $\mathbf{f}_2$, $\mathbf{f}_3$ orthogonal to it and each other, such that $\mathbf{f}_1 = \mathbf{f}_2 \times \mathbf{f}_3$, to define an RMF. Planar curves are also trivially RRMF curves, since the Darboux vector [18] for a plane curve (whose torsion satisfies $\tau \equiv 0$) reduces to $\boldsymbol{\omega} = \kappa\,\mathbf{b}$ — because this has no component in the direction of $\mathbf{t}$, the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is evidently rotation–minimizing.

Since we are primarily interested in RRMF curves that are proper space curves, we need a means to discount the degenerate cases of linear and planar curves. Consider first the case of degeneration to a straight line, characterized by zero curvature.

Lemma 3.1 *The curvature $\kappa(t)$ of the PH curve specified by* (4) *and* (5) *is identically zero if and only if $\alpha, \beta \in \mathbb{R}$ exist, such that*

$$p(t) \;=\; \alpha\, u(t) + \beta\, v(t) \qquad \textit{and} \qquad q(t) \;=\; \beta\, u(t) - \alpha\, v(t)\,. \tag{10}$$

Proof: ($\Longrightarrow$) The curvature of $\mathbf{r}(t)$ can be expressed [6] as

$$\kappa(t) \;=\; 2\,\frac{\sqrt{\rho(t)}}{\sigma^2(t)}\,,$$

where $\rho = (u'p - up' + v'q - vq')^2 + (u'q - uq' - v'p + vp')^2$. A straightforward calculation shows that, when (10) holds, we have $u'p - up' + v'q - vq' = u'q - uq' - v'p + vp' = 0$.

($\Longleftarrow$) Suppose that $\kappa(t) \equiv 0$. Since $\kappa(t) = |\mathbf{r}'(t) \times \mathbf{r}''(t)|/|\mathbf{r}'(t)|^3$, vanishing of the curvature implies that

$$\frac{x''(t)}{x'(t)} \;=\; \frac{y''(t)}{y'(t)} \;=\; \frac{z''(t)}{z'(t)}\,.$$

Since these ratios are the derivatives of $\ln x'(t)$, $\ln y'(t)$, $\ln z'(t)$ we have $\ln x'(t) = \ln y'(t) - \ln a = \ln z'(t) - \ln b$ and hence $y'(t) = ax'(t)$, $z'(t) = bx'(t)$ for positive real values a, b.

Now since $x' = u^2 + v^2 - p^2 - q^2$, $y' = 2uq + 2pv$, $z' = 2qv - 2up$, we have $x'^2 + y'^2 + z'^2 = (u^2 + v^2 + p^2 + q^2)^2 = x'^2(1 + a^2 + b^2)$. Without loss of generality, we may assume that $x' = c(u^2+v^2+p^2+q^2)$ where $c^{-1} = \sqrt{1 + a^2 + b^2}$, $0 < c < 1$. Combining $x' = u^2+v^2-p^2-q^2$ and $x' = c(u^2 + v^2 + p^2 + q^2)$ gives $u^2 + v^2 = \gamma(p^2 + q^2)$ and $x' = (\gamma - 1)(p^2 + q^2)$, where $\gamma = (1 + c)/(1 - c)$. We then have

$$\left[\begin{array}{c} y' \\ z' \end{array}\right] = \left[\begin{array}{cc} u & v \\ v & -u \end{array}\right] \left[\begin{array}{c} 2q \\ 2p \end{array}\right] = \left[\begin{array}{c} a\,x' \\ b\,x' \end{array}\right],$$

or

$$\left[\begin{array}{cc} u^2+v^2 & 0 \\ 0 & u^2+v^2 \end{array}\right] \left[\begin{array}{c} 2q \\ 2p \end{array}\right] = \left[\begin{array}{cc} u & v \\ v & -u \end{array}\right] \left[\begin{array}{c} a\,(\gamma-1)(p^2+q^2) \\ b\,(\gamma-1)(p^2+q^2) \end{array}\right].$$

Since $u^2 + v^2 = \gamma(p^2 + q^2)$, the preceding equation implies that (10) holds with

$$\alpha = -\frac{b(\gamma - 1)}{2\gamma} \qquad \text{and} \qquad \beta = \frac{a(\gamma - 1)}{2\gamma}.$$

Note that $\gamma = 1$ (i.e., $c = 0$) corresponds to the trivial case $\alpha = \beta = 0$. ∎

Lemma 3.1 generalizes results given in [9] for the degeneration of spatial PH cubics and quintics to straight lines, in terms of constraints on the quaternion coefficients, to curves of arbitrary degree. The Lemma 3.1 conditions (10) are equivalent to requiring the quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ to admit a factorization of the form

$$\mathcal{A}(t) = (1 + \alpha\,\mathbf{j} + \beta\,\mathbf{k})\,[\,u(t) + v(t)\,\mathbf{i}\,], \quad \alpha, \beta \in \mathbb{R}.$$

Consider now the case of planar PH curves (other than straight lines) — such curves have non–zero curvature, but zero torsion. However, the expression for the torsion of spatial PH curves is cumbersome and difficult to analyze. A simpler characterization of planar curves is in terms of the existence of a unit vector $\mathbf{n} = (\lambda, \mu, \nu)$ such that $\mathbf{n} \cdot \mathbf{r}'(t) \equiv 0$, i.e.,

$$[\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\lambda + 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mu + 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\nu \equiv 0\,.$$

Satisfaction of this condition corresponds to one of the following three cases (for brevity, we omit a complete analysis of their implications for u, v, p, q).

Case 1: If at least one of $x'(t)$, $y'(t)$, $z'(t)$ vanishes identically, the curve is trivially planar. Henceforth, we assume that none of $x'(t)$, $y'(t)$, $z'(t)$ is identically zero.

Case 2: If one of $x'(t)$, $y'(t)$, $z'(t)$ is a (constant) multiple of another — in this case we have $x'y'' - x''y' = 0$, or $y'z'' - y''z' = 0$, or $z'x'' - z''x' = 0$.

Case 3: In the generic case, with x', y', z' all non–vanishing and none a constant multiple of another, we have $x' = k\,y' + l\,z'$ for real non–zero values k, l. We then find that

$$(x'z'' - x''z')(y'z''' - y'''z') = (x'z''' - x'''z')(y'z'' - y''z')\,.$$

This is equivalent to $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' \equiv 0$, i.e., the torsion vanishes identically, and hence the curve $\mathbf{r}(t)$ is planar.

4 A general RRMF condition

Recall that the hodograph (5) defines an RRMF curve if and only if two relatively prime polynomials $a(t)$, $b(t)$ exist, such that the components $u(t)$, $v(t)$, $p(t)$, $q(t)$ of (4) satisfy the condition (8). Since quotients of the forms on the left and right in (8) appear frequently henceforth, we denote them by

$$[\,u,v,p,q\,] \,=\, \frac{uv'-u'v-pq'+p'q}{u^2+v^2+p^2+q^2} \qquad \text{and} \qquad [\,a,b\,] \,=\, \frac{ab'-a'b}{a^2+b^2}\,. \tag{11}$$

Evidently, an analysis of the structure of $[\,u,v,p,q\,]$ and $[\,a,b\,]$ is important to the problem of identifying RRMF curves. We begin by analyzing $[\,a,b\,]$.

4.1 Structure of the ratio $[\,a,b\,]$

Unless otherwise stated $a(t)$, $b(t)$ are non–zero polynomials in $\mathbb{R}[t]$ with $\gcd(a(t),b(t))=1$. Also, let $P(t)=a(t)b'(t)-a'(t)b(t)$, $Q(t)=a^2(t)+b^2(t)$ and assume that $\deg(Q)\geq 2$ and that $Q(t)$ is monic. The following Remark is the basis for most of the results in this section.

Remark 4.1 *Let* $F(t),G(t)\in\mathbb{R}[t]$ *satisfy* $\gcd(F,G)=\gcd(G,G')$*, and let* r *be a root of* $G(t)$ *of multiplicity* k*. If* $G(t)=(t-r)^k g(t)$ *and*

$$\frac{F(t)}{G(t)} \,=\, \frac{\alpha}{t-r} + \frac{f(t)}{g(t)}$$

is the expansion of $F(t)/G(t)$ *at* $t=r$*, then* $k\,F^{(k-1)}(r)=\alpha\,G^{(k)}(r)$*.*

Proof: We have $F(t)=\alpha\,(t-r)^{k-1}g(t)+(t-r)^k f(t)$. Since $F^{(k-1)}(r)=\alpha\,(k-1)!\,g(r)$ and $G^{(k)}(r)=k!\,g(r)$, the result follows. ■

Note that α is called the *residue* of F/G at $t=r$. Also, if r is non–real, the residue of F/G at $\overline{r}$ (the complex conjugate of r) is simply $\overline{\alpha}$.

Lemma 4.1 *If* a,b,P,Q *are as above, then* $\gcd(P,Q)=\gcd(Q,Q')$*, and* $\gcd(Q,Q')$ *divides* $\gcd(Q,a'^2+b'^2)$*.*

Proof: Let r be a root of Q of multiplicity k. Then $a^2(r)+b^2(r)=0$, and hence $a(r)=\pm\,\mathrm{i}\,b(r)$. When $k=1$, we have $P(r)=a(r)b'(r)-a'(r)b(r)=\pm\,\mathrm{i}\,[\,a(r)a'(r)+b(r)b'(r)\,]=$

$\pm\frac{1}{2}\,\mathrm{i}\,Q'(r)\neq 0$. Induction on k then reveals that $P^{(i-1)}(r)=Q^{(i)}(r)=0$ for $1\leq i\leq k-1$, and $P^{(k-1)}(r)=a(r)b^{(k)}(r)-b(r)a^{(k)}(r)=\pm\frac{1}{2}\,\mathrm{i}\,Q^{(k)}(r)\neq 0$.

Now we have $a^2(t)+b^2(t)=(t-r)^k f(t)$ and $a(t)a'(t)+b(t)b'(t)=(t-r)^{k-1}h(t)$, where $f(r)\,h(r)\neq 0$. Note also that $a(r)\,b(r)\neq 0$, since $\gcd(a,b)=1$. Consider the polynomial $z(t)=a(t)b(t)P(t)$. In view of the above, we obtain

$$\begin{aligned} z(t) &= b(t)b'(t)(t-r)^k f(t)-b^2(t)(t-r)^{k-1}h(t)\\ &= (t-r)^{k-1}[\,b(t)b'(t)(t-r)f(t)-b^2(t)h(t)\,]\,. \end{aligned}$$

Now $b(t)b'(t)(t-r)\,f(t)-\,b(t)^2h(t)\,|_{t=r}=-\,b^2(r)\,h(r)\neq 0$, so r is a root of $z(t)$ — and thus also of $P(t)$ — of multiplicity $k-1$. If $P(t)=(t-r)^{k-1}g(t)$, then $P^{(k-1)}(r)=(k-1)!\,g(r)$, and since $\frac{1}{2}\,Q^{(k)}(r)=(aa'+bb')^{(k-1)}(r)=(k-1)!\,h(r)$, we have $g^2(r)+h^2(r)=0$ because $P^{(k-1)}(r)=\pm\frac{1}{2}\,\mathrm{i}\,Q^{(k)}(r)$. Noting that

$$(ab'-a'b)^2\;+\;(aa'+bb')^2\;=\;(a^2+b^2)\,(a'^2+b'^2)\,,$$

the left–hand side is equal to $(t-r)^{2k-2}[\,g^2(t)+h^2(t)\,]$, which has r as a root of multiplicity $2k-1$ at least, since $g^2(r)+h^2(r)=0$. Thus, since r is a root of a^2+b^2 of multiplicity k, it must also be a root of $a'^2+b'^2$ of multiplicity $k-1$ at least. ■

Now let $r_1,\overline{r}_1,\ldots,r_m,\overline{r}_m$ be the distinct pairs of complex conjugate roots of $Q=a^2+b^2$, and $k_1,\ldots,k_m$ be their multiplicities. From Remark 4.1 and the proof of Lemma 4.1, we see that the residues of P/Q at r_j, $\overline{r}_j$ are $\pm\frac{1}{2}\,\mathrm{i}\,k_j$, $\mp\frac{1}{2}\,\mathrm{i}\,k_j$ respectively. Hence, the partial fraction decomposition of $[\,a,b\,]$ is given by

$$[\,a,b\,]\;=\;\frac{P}{Q}\;=\;\pm\frac{\mathrm{i}}{2}\sum_{j=1}^{m}\left[\frac{k_j}{t-r_j}-\frac{k_j}{t-\overline{r}_j}\right]. \qquad (12)$$

The terms of the sum in (12) can be simplified by expressing the roots in terms of real and imaginary parts as $r_j=\alpha_j+\mathrm{i}\,\beta_j$, so that

$$\frac{\mathrm{i}}{2}\left[\frac{k_j}{t-r_j}-\frac{k_j}{t-\overline{r}_j}\right]\;=\;\frac{-\,k_j\beta_j}{(t-\alpha_j)^2+\beta_j^2}\;=\;k_j\,[\,t-\alpha_j,\beta_j\,]\,.$$

Hence (12) can be written as

$$[\,a,b\,]\;=\;\pm\sum_{j=1}^{m}k_j\,[\,t-\alpha_j,\beta_j\,]\,.$$

Now for $k \geq 1$, let $r_1, \overline{r}_1, \ldots, r_k, \overline{r}_k$ be distinct complex conjugate pairs and $n_1, \ldots, n_k$ be positive integers. Writing $r_j = \alpha_j + \mathrm{i}\,\beta_j$ and $A_j = [\,t - \alpha_j, \beta_j\,]$, let s be a map that associates signs $s_1, \ldots, s_k$ to $A_1, \ldots, A_k$ where $s_j = \pm 1$ for all j. For each such s, we consider the sum

$$B_s \;=\; \sum_{j=1}^{k} s_j n_j A_j\,. \tag{13}$$

We then have:

Remark 4.2 *There exist real, relatively prime polynomials $a_k(t)$, $b_k(t)$ with $\deg(a_k^2 + b_k^2) = 2\sum_{i=1}^{k} n_i$ so that $B_s = [\,a_k, b_k\,]$. The roots of $a_k^2 + b_k^2$ are precisely $r_1, \overline{r}_1, \ldots, r_k, \overline{r}_k$ and their multiplicities are $n_1, \ldots, n_k$. In particular, $a_k^2 + b_k^2$ has distinct roots if $n_j = 1$ for all j.*

Proof: The proof is by induction on $N_k = \sum_{i=1}^{k} n_i$. If $N_k = 1$ and $r_1 = \alpha_1 + \mathrm{i}\,\beta_1$, we have $B_s = \pm A_1 = \pm [\,t - \alpha_1, \beta_1\,]$. The induction step follows from the observation that

$$\frac{ab' - a'b}{a^2 + b^2} + \frac{cd' - c'd}{c^2 + d^2} = \frac{ef' - e'f}{e^2 + f^2}, \qquad \frac{ab' - a'b}{a^2 + b^2} - \frac{cd' - c'd}{c^2 + d^2} = \frac{gh' - g'h}{g^2 + h^2}, \tag{14}$$

where we set $e = ac - bd$, $f = ad + bc$, and $g = ac + bd$, $h = bc - ad$, and the fact that $\gcd(e, f) = \gcd(g, h) = 1$ when $\gcd(a, b) = 1$ and $c = t - \alpha_j$, $d = \beta_j$. ■

The above indicates that $B_{s_1} \neq B_{s_2}$ for $s_1 \neq s_2$. Hence, for given distinct non real numbers $r_1, \overline{r}_1, \ldots, r_k, \overline{r}_k$ and positive integers $n_1, \ldots, n_k$ there are precisely 2^k different sums B_s.

Using the above results, we can now state the following.

Proposition 4.1 *Let $a(t)$, $b(t)$, $Q(t)$ be as specified above. Then we have*

$$[\,a, b\,] \;=\; \sum_{j=0}^{m} s_j k_j A_j \tag{15}$$

for distinct non–real complex numbers $r_1, \overline{r}_1, \ldots, r_m, \overline{r}_m$ positive integers $k_1, \ldots, k_m$ and an appropriate sign map s. Moreover, the representation (15) *of $[\,a, b\,]$ is unique. Conversely, for any given sum of the form $S = s_1 k_1 A_1 + \cdots + s_m k_m A_m$, there exist relatively prime polynomials $a(t), b(t) \in \mathbb{R}[t]$ such that $S = [\,a, b\,]$.*

Proof: The representation (15) of $[\,a, b\,]$ is simply its partial fraction decomposition, and is thus unique. Also, from Remark 4.2, there exist polynomials $a(t), b(t) \in \mathbb{R}[t]$ such that $S = [\,a, b\,]$ and $\deg(a^2 + b^2) = 2\sum_{i=1}^{m} k_i$. ■

Now any (strictly) positive real polynomial $f(t)$ can be expressed as a sum of the squares of two real polynomials, $f^2(t) = g^2(t) + h^2(t)$, in infinitely many ways. We are concerned here with the number of different ways in which the quotient $[\,a,b\,] = (ab' - a'b)/(a^2 + b^2)$ can be expressed. Setting $Q(t) = a^2(t) + b^2(t)$ for given $a(t), b(t) \in \mathbb{R}[t]$ with $\gcd(a,b) = 1$, suppose we also have $Q(t) = c^2(t) + d^2(t)$ for $c(t), d(t) \in \mathbb{R}[t]$ with $\gcd(c,d) = 1$. We then say that (a,b) and (c,d) are *similar*, and write $(a,b) \sim (c,d)$, if we have $[\,a,b\,] = [\,c,d\,]$.

We now identify the conditions under which the equivalence relation $(a,b) \sim (c,d)$ holds.

Proposition 4.2 *Let $Q = a^2 + b^2 = c^2 + d^2$. Then we have*

1. *$(a,b) \sim (c,d)$ if and only if $c + \mathrm{i}\,d = \mathbf{z}\,(a + \mathrm{i}\,b)$ where $\mathbf{z}$ is a unit complex number.*

2. *The number of distinct polynomial pairs equivalent to (a,b) is 2^m, where $2m$ is the number of distinct roots of $Q = a^2 + b^2$.*

Proof: 1. By observing that

$$\frac{ab' - a'b}{a^2 + b^2} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}(b/a) \qquad \text{and} \qquad \frac{cd' - c'd}{c^2 + d^2} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}(d/c)\,,$$

we have

$$\tan^{-1}(d/c) \;=\; \tan^{-1}(b/a) \;+\; \phi\,,$$

ϕ being an arbitrary integration constant. Taking the tangent of both sides and clearing denominators then yields

$$\frac{d}{c} \;=\; \frac{b + \tau\,a}{a - \tau\,b}\,,$$

where $\tau = \tan\phi$. Since $d(a - \tau\,b) = c(b + \tau\,a)$ and $\gcd(c,d) = \gcd(a - \tau\,b, b + \tau\,a) = 1$, there exists $\lambda \in \mathbb{R}$ so that $c(t) = \lambda\,[\,a(t) - \tau\,b(t)\,]$ and $d(t) = \lambda\,[\,\tau\,a(t) + b(t)\,]$. Thus, $c + \mathrm{i}\,d = \lambda(1 + \mathrm{i}\,\tau)(a + \mathrm{i}\,b)$. Note that $|\,\lambda(1 + \mathrm{i}\,\tau)\,| = 1$, since $a^2 + b^2 = c^2 + d^2$. The converse is a straightforward calculation.

2. Proposition 4.1 shows that $[\,a,b\,]$ is equal, for appropriate $k_1, \ldots, k_m$ and s, to the sum

$$S \;=\; \sum_{j=1}^{m} s_j k_j A_j\,,$$

and according to Remark 4.2 there are precisely 2^m such sums. ∎

4.2 RRMF conditions

The preceding section described the structure of $[\,a,b\,]$ in great detail. A similar analysis of $[\,u,v,p,q\,]$ seems to be a much more difficult task. Nevertheless, the results of the previous section can be used to identify useful constraints for PH curves to have rational RMFs.

Now for $u,v,p,q \in \mathbb{R}[t]$ with $\gcd(u,v,p,q)=1$ let

$$w \;=\; uv' - u'v - pq' + p'q \tag{16}$$

be the numerator on the left in (8), and $\sigma = u^2+v^2+p^2+q^2$ be its denominator as in (3). Also, let $\eta = \frac{1}{4}\sigma'^2 + w^2$ be defined in terms of them by

$$\eta \;=\; (uu'+vv'+pp'+qq')^2 \;+\; (uv'-u'v-pq'+p'q)^2\,. \tag{17}$$

We may assume, without loss of generality, that σ is monic, and $\deg(\sigma)\geq 2$.

Question 1 *For u,v,p,q,w,σ as above, under what conditions is (8) satisfied for $a,b \in \mathbb{R}[t]$ with $\gcd(a,b)=1$?*

The following provides an answer to this question under the assumption that the quotient w/σ is *proper* — i.e., that $\gcd(\sigma,w)=1$.

Proposition 4.3 *Let the polynomials $u(t),v(t),p(t),q(t)\in\mathbb{R}[t]$ satisfy $\gcd(u,v,p,q)=1$, and let $\sigma(t)$ and $w(t)$ defined by (3) and (16) satisfy $\gcd(\sigma,w)=1$. Then (8) is satisfied by $a(t),b(t)\in\mathbb{R}[t]$ with $\gcd(a,b)=1$ if and only if the polynomial $\eta(t)$ defined by (17) is divisible by $\sigma(t)$, i.e., a polynomial $h(t)\in\mathbb{R}[t]$ exists such that*

$$\eta(t) \;=\; \sigma(t)\,h(t)\,. \tag{18}$$

Proof: Suppose (18) is satisfied, and let r be a root of σ of multiplicity k. Then since $\eta = \frac{1}{4}\sigma'^2 + w^2$, we must have $w(r)=0$ if $k\geq 2$, contradicting the fact that $\gcd(\sigma,w)=1$. Thus, σ has only simple roots. Now from Remark 4.1, the residue α of w/σ at r satisfies $w(r)=\alpha\,\sigma'(r)$. Also, since (18) must vanish at each root of σ, we have $\eta(r)=\frac{1}{4}\sigma'^2(r) + w^2(r)=0$, or $2w(r)=\pm\,\mathrm{i}\,\sigma'(r)$. Hence, $\alpha=\pm\frac{1}{2}\mathrm{i}$ and w/σ is of the form (13). Therefore, from Remark 4.2, relatively prime polynomials a,b exist such that $w/\sigma=[\,a,b\,]$.

Conversely, suppose (8) is satisfied, and let r be a root of σ of multiplicity m. Since $w(r) \neq 0$ and w/σ is equal to $[\,a,b\,]$ we must have $m=1$ from (12). Hence, $\gcd(\sigma,\sigma')=1$. In that case, it follows from (15) that $[\,u,v,p,q\,] = s_1A_1+s_2A_2+\cdots+s_kA_k$ where $A_j=[\,t-\alpha_j,\beta_j\,]$ and $A_i \neq A_j$. From Remark 4.2, polynomials $f,g \in \mathbb{R}[t]$ exist such that $\sigma = f^2+g^2$. Thus, $w = fg'-f'g$ and we then have

$$\eta \,=\, \tfrac{1}{4}\,\sigma'^2 + w^2 \,=\, (ff'+gg')^2 + (fg'-f'g)^2 \,=\, (f^2+g^2)(f'^2+g'^2) \,=\, \sigma\,h\,,$$

where $h = f'^2+g'^2$. Hence (18) is also a necessary condition for satisfaction of (8). ∎

Note that, in terms of the quaternion representation of spatial PH curves defined by (4)–(5), the terms appearing in (17) can be expressed as

$$uu'+vv'+pp'+qq' \,=\, \mathrm{scal}(\mathcal{A}\,\mathcal{A}'^*)\,, \quad uv'-u'v-pq'+p'q \,=\, \mathrm{scal}(\mathcal{A}\,\mathbf{i}\,\mathcal{A}'^*)\,,$$

and since $\sigma = |\mathcal{A}|^2$, the RRMF condition (18) can be written as

$$[\,\mathrm{scal}(\mathcal{A}(t)\,\mathcal{A}'^*(t))\,]^2 \,+\, [\,\mathrm{scal}(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t))\,]^2 \,=\, |\mathcal{A}(t)|^2\,h(t)\,. \qquad (19)$$

We now relax the requirement $\gcd(\sigma,w)=1$, but slightly modify Question 1 to avoid some very pathological cases.

Question 2 *For u,v,p,q,w,σ as above, under what conditions is* (8) *satisfied for $a,b \in \mathbb{R}[t]$ with* $\gcd(a,b)=1$ and $\sigma = a^2+b^2$*?*

The answer to this question is similar to the previous one, as follows.

Proposition 4.4 *Let the polynomials $u(t),v(t),p(t),q(t) \in \mathbb{R}[t]$ satisfy* $\gcd(u,v,p,q)=1$, *and let $f(t)$ be defined by $f=\gcd(\sigma,\sigma')$. Then* (8) *is satisfied by polynomials $a(t),b(t) \in \mathbb{R}[t]$ with $\gcd(a,b)=1$ and $a^2+b^2=\sigma$ if only if the polynomial $\eta(t)$ defined by* (17) *is divisible by $\sigma(t)\,f(t)$, i.e., a polynomial $g(t) \in \mathbb{R}[t]$ exists, such that*

$$\eta(t) \,=\, \sigma(t)\,f(t)\,g(t)\,. \qquad (20)$$

Proof: Suppose (20) is satisfied, so that

$$\eta \,=\, \tfrac{1}{4}\,\sigma'^2 + w^2 \,=\, \sigma f g\,. \qquad (21)$$

Now (21) implies that f divides w, since f^2 divides w^2, and we can write $w = fA$, $\sigma' = fB$ for $A, B \in \mathbb{R}[t]$. If r is a root of σ of multiplicity m, we have $w^{(m-1)}(r) = f^{(m-1)}(r)A(r)$ and $\sigma^{(m)}(r) = f^{(m-1)}(r)B(r)$. Moreover, $A^2(r) + \frac{1}{4}B^2(r) = 0$ or $A(r) = \pm\frac{1}{2}\,\mathrm{i}\,B(r)$, since $\sigma fg/f^2$ vanishes at $t = r$. Thus $w^{(m-1)}(r) = \pm\frac{1}{2}\,\mathrm{i}\,\sigma^{(m)}(r) \neq 0$, and this implies that $\gcd(w,\sigma) = \gcd(\sigma,\sigma')$. Now from Remark 4.1 we see that the residue of w/σ at r is $\pm\frac{1}{2}\,\mathrm{i}\,m$, so w/σ is of the form (13) and is thus equal to $[\,a, b\,]$ for $a, b \in \mathbb{R}[t]$.

Conversely, suppose (8) is satisfied with $\sigma = u^2 + v^2 + p^2 + q^2 = a^2 + b^2$. We then have $\sigma' = 2(uu' + vv' + pp' + qq') = 2(aa' + bb')$, and (8) implies that $w = uv' - u'v - pq' + p'q = ab' - a'b$. Hence,

$$(uv'-u'v-pq'+p'q)^2+(uu'+vv'+pp'+qq')^2 = (ab'-a'b)^2+(aa'+bb')^2 = (a^2+b^2)(a'^2+b'^2)\,.$$

Thus $\eta = \sigma(a'^2+b'^2)$. But according to Lemma 4.1, $a'^2+b'^2$ is divisible by f so $a'^2+b'^2 = fg$ for some $g \in \mathbb{R}[t]$. Thus (20) is also necessary for satisfaction of (8) with $\sigma = a^2 + b^2$. ■

Propositions 4.3 and 4.4 show that satisfaction of the RRMF condition (8) is expressible in terms of divisibility of the polynomial $\eta(t)$, defined by (17), by the parametric speed $\sigma(t)$ of the PH curve, specified by (3). These results can be alternatively expressed in terms of the polynomial

$$\rho \,=\, (up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2\,. \tag{22}$$

Remark 4.3 *$\eta(t)$ is divisible by $\sigma(t)$ if and only if $\rho(t)$ is divisible by $\sigma(t)$.*

Proof: From the definitions (3), (17), and (22) of σ, η, and ρ, one can verify that

$$\eta \,+\, \rho \,=\, \sigma\,(u'^2 + v'^2 + p'^2 + q'^2)\,.$$

Since σ divides the sum of η and ρ, if it divides one of them, it must divide the other. ■

The polynomial $\rho(t)$ plays a more established role in the theory of spatial PH curves. For example, the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ of the first and second derivatives of such curves satisfies [6] the condition[3]

$$|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|^2 \,=\, 4\,\sigma^2(t)\,\rho(t)\,,$$

[3]Note that, in [6] and earlier papers, the factor 4 in this relation is absorbed into $\rho(t)$.

and $4\,\rho(t)$ can be identified with $|\mathbf{r}''(t)|^2-\sigma'^2(t)$. The theory of *double PH curves* [9, 10, 19] is based on the requirement that $|\mathbf{r}'(t)\times\mathbf{r}''(t)|$, as well as $|\mathbf{r}'(t)|$, must be a polynomial in t — i.e., $\rho(t)$ must be a perfect square. Such curves have rational Frenet frames and curvatures. Note also that $\rho(t)$ admits a very simple expression [6] in terms of the complex polynomials (6) that define the Hopf map form (7) of spatial PH curves, namely

$$\rho(t) \;=\; |\,\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t)-\boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)\,|^2\,. \tag{23}$$

Remark 4.4 *If* $\deg(u,v,p,q)=m$, *we have* $\deg(\sigma)=2m$ *and* $\deg(\rho)=4m-4$ *due to a cancellation of highest–order terms in* $up'-u'p$ *and analogous expressions. Hence,* $\deg(\sigma)$ *is greater than, equal to, or less than* $\deg(\rho)$ *when* $m=1$, $m=2$, *and* $m\geq 3$, *respectively. Since* σ *can divide* ρ *only if* $\deg(\sigma)\leq\deg(\rho)$, *this clarifies why no proper RRMF cubics* $(m=1)$ *exist [13], and the lowest–order proper RRMF curves are [7, 8] quintics* $(m=2)$.

Note that the preceding observation could not have been made on the basis of divisibility of η by σ, since in general $\deg(\eta)=4m-2$ (no cancellation of highest–order terms occurs in $uu'+vv'+pp'+qq'$). Thus it seems preferable, on several accounts, to express the RRMF conditions of Propositions 4.3 and 4.4 in terms of the divisibility of ρ, rather than η, by σ.

Example 1 *In the Hopf map representation* (7), *a spatial PH quintic is specified by two quadratic complex polynomials*

$$\boldsymbol{\alpha}(t) \;=\; \boldsymbol{\alpha}_0\,b_0^2(t)+\boldsymbol{\alpha}_1\,b_1^2(t)+\boldsymbol{\alpha}_2\,b_2^2(t)\,,\quad \boldsymbol{\beta}(t) \;=\; \boldsymbol{\beta}_0\,b_0^2(t)+\boldsymbol{\beta}_1\,b_1^2(t)+\boldsymbol{\beta}_2\,b_2^2(t)\,,$$

where

$$b_i^n(t) \;=\; \binom{n}{i}(1-t)^{n-i}t^i\,,\quad i=0,\ldots,n$$

is the Bernstein basis of degree n *on* $t\in[\,0,1\,]$. *In this case,* $\rho(t)=|\,\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t)-\boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)\,|^2$ *and* $\sigma(t)=|\boldsymbol{\alpha}(t)|^2+|\boldsymbol{\beta}(t)|^2$ *are both quartic, and the requirement that the latter should divide the former implies that they are proportional to each other. It was shown in [7] that, for quintics, satisfaction of the constraints*

$$\mathrm{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2-\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2) \;=\; |\boldsymbol{\alpha}_1|^2-|\boldsymbol{\beta}_1|^2 \qquad \textit{and} \qquad \boldsymbol{\alpha}_0\overline{\boldsymbol{\beta}}_2+\boldsymbol{\alpha}_2\overline{\boldsymbol{\beta}}_0 \;=\; 2\,\boldsymbol{\alpha}_1\overline{\boldsymbol{\beta}}_1$$

is a sufficient and necessary condition for an RRMF curve. Thus, for example, the values

$$\boldsymbol{\alpha}_0=1+\mathrm{i}\,,\;\; \boldsymbol{\alpha}_1=\sqrt{5}\,\mathrm{i}\,,\;\; \boldsymbol{\alpha}_2=2-\mathrm{i}\,,\;\; \boldsymbol{\beta}_0=-1+\mathrm{i},\;\; \boldsymbol{\beta}_1=\frac{1-2\,\mathrm{i}}{\sqrt{5}}\,,\;\; \boldsymbol{\beta}_2=1-2\,\mathrm{i}$$

satisfying these constraints define an RRMF quintic. For this curve, we obtain

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) \,=\, \frac{8(2+\mathrm{i})}{\sqrt{5}}\, b_0^2(t) \,+\, 4(1-\mathrm{i})\, b_1^2(t) \,+\, 4\sqrt{5}\,(1+\mathrm{i})\, b_2^2(t)\,,$$

and

$$|\boldsymbol{\alpha}(t)|^2 \,=\, 2\,b_0^4(t) \,+\, \sqrt{5}\,b_1^4(t) \,+\, \frac{11}{3}\,b_2^4(t) \,-\, \sqrt{5}\,b_3^4(t) \,+\, 5\,b_4^4(t)\,,$$
$$|\boldsymbol{\beta}(t)|^2 \,=\, 2\,b_0^4(t) \,-\, \frac{3}{\sqrt{5}}\,b_1^4(t) \,-\, \frac{1}{3}\,b_2^4(t) \,+\, \sqrt{5}\,b_3^4(t) \,+\, 5\,b_4^4(t)\,,$$

and hence we find that $|\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)|^2 = 16\,(|\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2)$*, where*

$$|\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2 \,=\, 4\,b_0^4(t) \,+\, \frac{2}{\sqrt{5}}\,b_1^4(t) \,+\, \frac{10}{3}\,b_2^2(t) \,+\, 0\,b_3^4(t) + 10\,b_4^4(t)\,.$$

Figure 1 compares the Frenet and rational rotation–minimizing frames on this curve.

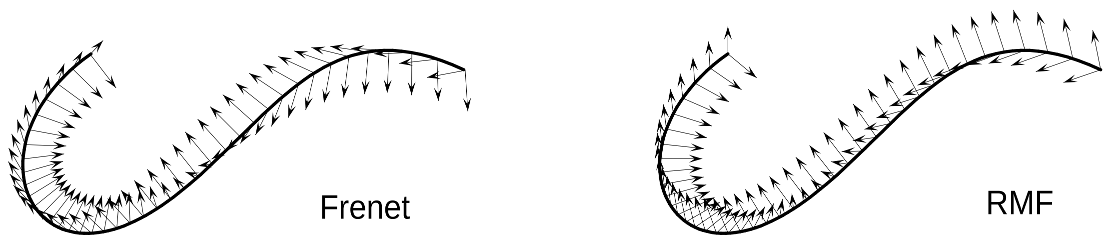

Figure 1: Comparison of Frenet frame (left) and rational rotation–minimizing frame (right) on the quintic RRMF curve of Example 1. For clarity, only the two normal–plane basis vectors are shown — the unit tangent vector (common to both frames) is omitted.

5 RRMF curves of any degree

The preceding methods can be used to demonstrate the existence of RRMF curves of any (odd) degree, $n = 2m+1$. The aim here is not to develop a complete characterization of higher–order RRMF curves, but simply to show that representative examples can be easily constructed. The argument is by induction on m — since the cases $m = 1, 2$ have already been well–studied [8, 7, 13] we focus on $m \geq 3$.

Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be an RRMF curve of degree $2m+1$, specified by (4) and (5). Then we have $[\,u, v, p, q\,] = [\,a, b\,]$ for some relatively prime polynomials a, b. Let $r = \alpha + \mathrm{i}\,\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, and set $A = t - \alpha, B = \beta$. The induction step follows from the following result, which may be verified by straightforward calculation.

Proposition 5.1 *Setting* $u_1 = uA - vB$, $v_1 = uB + vA$, $p_1 = pA + qB$, $q_1 = -pB + qA$, $a_1 = aA - bB$, $b_1 = aB + bA$ *for* u, v, p, q, a, b, A, B *as specified above, we have*

$$[\, u_1, v_1, p_1, q_1 \,] \;=\; [\, a_1, b_1 \,] \,.$$

Since $\deg(u_1, v_1, p_1, q_1) = m + 1$ when $\deg(u, v, p, q) = m$, this generates RRMF curves of degree $n + 2 = 2(m + 1) + 1$ from a given RRMF curve of degree $n = 2m + 1$. However, although it provides a theoretical and practical means for constructing RRMF curves of any degree, it is far from identifying all such curves of a given degree.

Finally, note that when (8) is satisfied we must have $\deg(a^2 + b^2) \le \deg(\sigma)$, as follows.

Remark 5.1 *Let* u, v, p, q, σ *be as before, and suppose that* $[\, u, v, p, q \,] = [\, a, b \,]$ *for non–zero* $a, b \in \mathbb{R}[t]$ *with* $\gcd(a, b) = 1$. *Then*

1. *if we identify* (a, b) *with* $a + \mathrm{i}\, b$, *it is unique up to multiplication by a complex number;*

2. $\deg(a^2 + b^2) \le \deg(\sigma)$.

Proof: 1. Suppose (c, d) satisfy $[\, u, v, p, q \,] = [\, c, d \,]$ with $\gcd(c, d) = 1$. By Proposition 4.2, there exist $\lambda, \tau \in \mathbb{R}$ such that $c(t) = \lambda\,[\, a(t) - \tau\, b(t) \,]$, $d(t) = \lambda\,[\, \tau\, a(t) + b(t) \,]$. Then we have $c + \mathrm{i}\, d = \lambda(1 + \mathrm{i}\,\tau)(a + \mathrm{i}\, b)$, as required.

2. Let $r_1, \overline{r}_1, \ldots, r_k, \overline{r}_k$ be the distinct (non–real) roots of σ and $n_1, \ldots, n_k$ be their respective multiplicities. In view of (8) and (12) we see that $r_1, \overline{r}_1, \ldots, r_k, \overline{r}_k$ are the distinct (non–real) roots of $a^2 + b^2$ with multiplicities $n'_1, \ldots, n'_k$ satisfying $n'_i \le n_i$. Thus,

$$\deg(a^2 + b^2) \;=\; 2 \sum_{i=1}^{k} n'_i \;\le\; 2 \sum_{i=1}^{k} n_i \;=\; \deg(\sigma) \,.$$

■

The following example shows that there are cases where the strict inequality in Item 2 above is satisfied.

Example 2 *Let* $u = t$, $v = t^2$, $p = t$, $q = 1$, $a = 1$, $b = t$. *Then* $\deg(\sigma) = 4$, *and since*

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} \;=\; \frac{t^2 + 1}{(t^2 + 1)^2} \;=\; \frac{1}{t^2 + 1} \;=\; \frac{ab' - a'b}{a^2 + b^2} \,,$$

we see that $\deg(a^2 + b^2) = 2 < \deg(\sigma) = 4$. *Note, however, that this curve is planar since* $z' \equiv 0$, *and has a non–primitive hodograph with* $\gcd(x', y') = t^2 + 1$.

6 Closure

For a spatial PH curve defined through the quaternion form (5) or Hopf map form (7) by the quaternion polynomial (4) or pair of complex polynomials (6), a sufficient and necessary condition for the curve to admit a rational rotation–minimizing frame has been identified. Namely, either of the polynomials $\eta(t)$ or $\rho(t)$ defined by (17) or (22) must be divisible by the parametric speed $\sigma(t) = |\mathcal{A}(t)|^2 = |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2$ defined by (3).

Since $\rho(t)$ is of lower degree than $\eta(t)$, and more intimately connected to the established theory of spatial PH curves, it is generally preferable to express the RRMF condition in terms of the divisibility of $\rho(t)$, rather than $\eta(t)$, by $\sigma(t)$. This yields a simple explanation for the non–existence [13] of (non–planar) RRMF cubics, and a novel interpretation of the known characterization [7] for RRMF quintics: the quartic polynomials $\rho(t)$ and $\sigma(t)$ must be proportional. It is also noteworthy that $\rho(t)$ admits the exceedingly compact expression (23) in terms of the Hopf map representation (6)–(7).

Unlike previous characterizations [8, 7, 13] that focus on cubics and quintics, the criterion identified herein applies to RRMF curves of arbitrary (odd) degree. Although expressing the RRMF constraint as a divisibility requirement for certain polynomials is perhaps more existential than constructive in nature, it offers the theoretical basis for more constructive and algorithmic approaches, that we hope to explore in subsequent studies.

Finally, note that the results obtained herein apply to satisfaction of the RRMF condition (8) in the case where the denominators on the left and right hand sides are of equal degree.

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