A complete classification of quintic space curves with rational rotation–minimizing frames

Rida T. Farouki*
Department of Mechanical and Aerospace Engineering,
University of California, Davis, CA 95616, USA

Takis Sakkalis
Mathematics Laboratory, Agricultural University of Athens,
75 Iera Odos, Athens 11855, GREECE

Abstract

An adapted orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on a space curve $\mathbf{r}(t)$, where $\mathbf{f}_1 = \mathbf{r}'/|\mathbf{r}'|$ is the curve tangent, is *rotation–minimizing* if its angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$, i.e., the normal–plane vectors $\mathbf{f}_2, \mathbf{f}_3$ exhibit no instantaneous rotation about $\mathbf{f}_1$. The simplest space curves with *rational* rotation–minimizing frames (RRMF curves) form a subset of the quintic spatial *Pythagorean–hodograph* (PH) *curves*, identified by certain non–linear constraints on the curve coefficients. Such curves are useful in motion planning, swept surface constructions, computer animation, robotics, and related fields. The condition that identifies the RRMF quintics as a subset of the spatial PH quintics requires a rational expression in four quadratic polynomials $u(t), v(t), p(t), q(t)$ and their derivatives to be reducible to an analogous expression in just two polynomials $a(t), b(t)$. This condition has been analyzed, thus far, in the case where $a(t), b(t)$ are also quadratic, the corresponding solutions being called *Class I* RRMF quintics. The present study extends these prior results to provide a complete categorization of all possible PH quintic solutions to the RRMF condition. A family of *Class II* RRMF quintics is thereby newly identified, that correspond to the case where $a(t), b(t)$ are linear. Modulo scaling/rotation transformations, Class II curves have five degrees of freedom, as with the Class I curves. Although Class II curves have rational RMFs that are only of degree 6 — as compared to degree 8 for Class I curves — their algebraic characterization is more involved than for the latter. Computed examples are used to illustrate the construction and properties of this new class of RRMF quintics. A novel approach to generating RRMF quintics, based on the sum–of–four–squares decomposition of positive real polynomials, is also introduced and briefly discussed.

Keywords: rotation–minimizing frames; Pythagorean–hodograph curves;
complex numbers; quaternions; Hopf map; polynomial identities.

e–mail: farouki@ucdavis.edu, stp@aua.gr

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*Corresponding author (phone: 530-752-1779, fax: 530-752-4158)

1 Introduction

Let $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be an *adapted* orthonormal frame on a space curve $\mathbf{r}(t)$, such that $\mathbf{f}_1$ coincides with the curve tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ while $\mathbf{f}_2, \mathbf{f}_3$ span the normal plane at each curve point. The variation of such a frame may be specified by its angular velocity $\boldsymbol{\omega}$ through the differential equations

$$\mathbf{f}_1' \,=\, \boldsymbol{\omega} \times \mathbf{f}_1\,, \qquad \mathbf{f}_2' \,=\, \boldsymbol{\omega} \times \mathbf{f}_2\,, \qquad \mathbf{f}_3' \,=\, \boldsymbol{\omega} \times \mathbf{f}_3\,,$$

and the characteristic property of a *rotation–minimizing* frame is that the angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$, i.e., $\mathbf{f}_2, \mathbf{f}_3$ have no instantaneous rotation about $\mathbf{f}_1$. Such frames are of great interest in applications concerned with controlling the orientation of a rigid body along a spatial trajectory — for example, in swept surface constructions, computer animation, and robot path planning [1, 7, 8, 12, 13, 14, 15, 16].

Recent studies [5, 6, 8, 9, 10] have established the possibility of constructing *rational* rotation–minimizing frames on a special class of space curves of minimum degree 5 — the so–called *RRMF curves.* Such curves are necessarily Pythagorean–hodograph (PH) curves [4], since only PH curves admit rational unit tangents. The RRMF curves can thus be characterized by the identification of constraints on the coefficients of PH curves, that are sufficient and necessary for a rational RMF.

A spatial PH curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is characterized [4] by the property that its derivative components satisfy, for some polynomial $\sigma(t)$, the Pythagorean condition

$$x'^2(t) + y'^2(t) + z'^2(t) \,=\, \sigma^2(t)\,. \tag{1}$$

The *quaternion* and *Hopf map* forms [3] are two convenient models for the construction of spatial PH curves. The former generates a Pythagorean hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t)$ from a quaternion polynomial[1]

$$\mathcal{A}(t) \,=\, u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}\,, \tag{2}$$

and its conjugate $\mathcal{A}^*(t) = u(t) - v(t)\,\mathbf{i} - p(t)\,\mathbf{j} - q(t)\,\mathbf{k}$ through the product

$$\begin{aligned} \mathbf{r}'(t) \;&=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \,=\, [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} \\ &\qquad +\, 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \,+\, 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}\,. \end{aligned} \tag{3}$$

The latter employs two complex polynomials

$$\boldsymbol{\alpha}(t) \,=\, u(t) + \mathrm{i}\,v(t)\,, \quad \boldsymbol{\beta}(t) \,=\, q(t) + \mathrm{i}\,p(t) \tag{4}$$

to generate a Pythagorean hodograph through the expression

$$\mathbf{r}'(t) \,=\, (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)))\,. \tag{5}$$

The equivalence of (3) and (5) is seen by setting $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$, and identifying the imaginary unit i with the quaternion element $\mathbf{i}$. See [4] for a thorough treatment of these two representations. The parametric speed $\sigma(t) = |\mathbf{r}'(t)|$ of the PH curve $\mathbf{r}(t)$ defined by integrating $\mathbf{r}'(t)$ — i.e., the derivative of its arc length s with respect to the parameter t — is the polynomial

$$\sigma(t) \,=\, |\mathcal{A}(t)|^2 \,=\, |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2 \,=\, u^2(t) + v^2(t) + p^2(t) + q^2(t)\,. \tag{6}$$

The *Euler–Rodrigues frame* (ERF) is a rational adapted frame, defined [2] on any spatial PH curve by

$$(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)) \,=\, \frac{(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t), \mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t), \mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t))}{|\mathcal{A}(t)|^2}\,, \tag{7}$$

that is a useful reference for identifying rational RMFs. Here, $\mathbf{e}_1(t)$ is the curve tangent while $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$ span the normal plane. The ERF is given explicitly in terms of the polynomials $u(t), v(t), p(t), q(t)$ as

$$\begin{aligned} \mathbf{e}_1 \;&=\; \frac{(u^2 + v^2 - p^2 - q^2)\,\mathbf{i} \,+\, 2(uq + vp)\,\mathbf{j} \,+\, 2(vq - up)\,\mathbf{k}}{u^2 + v^2 + p^2 + q^2}\,, \\ \mathbf{e}_2 \;&=\; \frac{2(vp - uq)\,\mathbf{i} \,+\, (u^2 - v^2 + p^2 - q^2)\,\mathbf{j} \,+\, 2(uv + pq)\,\mathbf{k}}{u^2 + v^2 + p^2 + q^2}\,, \\ \mathbf{e}_3 \;&=\; \frac{2(up + vq)\,\mathbf{i} \,+\, 2(pq - uv)\,\mathbf{j} \,+\, (u^2 - v^2 - p^2 + q^2)\,\mathbf{k}}{u^2 + v^2 + p^2 + q^2}\,. \end{aligned} \tag{8}$$

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[1] Calligraphic characters such as $\mathcal{A}$ denote quaternions, their scalar and vector parts being indicated by $\mathrm{scal}(\mathcal{A})$ and $\mathrm{vect}(\mathcal{A})$. Bold symbols denote complex numbers or vectors in $\mathbb{R}^3$ — the meaning should be clear from the context.

Now if the PH curve defined by (3) or (5) admits a rational RMF $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ then $\mathbf{e}_1 = \mathbf{f}_1$ is the curve tangent, and the normal–plane vectors $\mathbf{f}_2(t), \mathbf{f}_3(t)$ must be obtainable from the ERF normal–plane vectors $\mathbf{e}_2(t), \mathbf{e}_3(t)$ by a rational rotation — i.e., for relatively prime polynomials $a(t), b(t)$ we must have

$$\left[\begin{array}{c} \mathbf{f}_2(t) \\ \mathbf{f}_3(t) \end{array} \right] = \frac{1}{a^2(t)+b^2(t)} \left[\begin{array}{cc} a^2(t)-b^2(t) & -\,2\,a(t)b(t) \\ 2\,a(t)b(t) & a^2(t)-b^2(t) \end{array} \right] \left[\begin{array}{c} \mathbf{e}_2(t) \\ \mathbf{e}_3(t) \end{array} \right] . \tag{9}$$

This is equivalent [11] to the requirement that

$$\frac{uv' - u'v - pq' + p'q}{u^2+v^2+p^2+q^2} = \frac{ab'-a'b}{a^2+b^2} \tag{10}$$

for relatively prime polynomials $a(t), b(t)$. The expression on the left is just the component $\omega_1 = \boldsymbol{\omega}\cdot\mathbf{t}$ of the ERF angular velocity $\boldsymbol{\omega}$ in the direction of $\mathbf{e}_1 = \mathbf{f}_1$, while that on the right is the angular velocity of the normal–plane rotation (9) that maps $\mathbf{e}_2, \mathbf{e}_3$ onto $\mathbf{f}_2, \mathbf{f}_3$. Thus, the condition (10) requires the existence of a rational normal–plane rotation that exactly cancels the ω_1 component of the ERF angular velocity. In terms of the Hopf map representation, condition (10) is equivalent to requiring the existence of a complex polynomial $\mathbf{w}(t) = a(t) + \mathrm{i}\,b(t)$, with $\gcd(a(t), b(t)) = 1$, such that

$$\frac{\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{\mathrm{Im}(\overline{\mathbf{w}}\mathbf{w}')}{|\mathbf{w}|^2} . \tag{11}$$

The simplest non–planar curves with rational RMFs are quintics [5, 6, 11]. To define a PH quintic through (3) or (5), the polynomials $u(t), v(t), p(t), q(t)$ must be quadratic. Satisfaction of the RRMF condition (10) by PH quintics has thus far been considered only in the case where $a(t), b(t)$ are assumed to also be quadratic. In this case, simple constraints on the coefficients of the quaternion polynomial (2) or the complex polynomials (4) have been identified [5] that are sufficient and necessary for a rational RMF. These were also shown [9] to be equivalent to a certain polynomial divisibility condition. The simplicity of the constraints identifying these RRMF quintics, together with (modulo scaling/rotation transformations) their five residual degrees of freedom, facilitates development of algorithms [8] for the design of rational rotation–minimizing rigid–body motions by the interpolation of initial/final positions and orientations.

This study extends and completes these prior results by enumerating a complete categorization of all solutions to (10) by PH quintics — i.e., when $u(t), v(t), p(t), q(t)$ are quadratic but $a(t), b(t)$ are of *unrestricted* degree. The most important outcome of this classification is the identification of a novel non–trivial family of RRMF quintics that satisfy (10) with $a(t), b(t)$ linear rather than quadratic. Because the rational normal–plane rotation in (9) is quadratic rather than quartic, this new class of RRMF quintics admit rational RMFs of lower degree than those for which $a(t), b(t)$ are quadratic, and they also have five essential degrees of freedom. However, their algebraic characterization appears to be inherently more complicated.

The plan for this paper is as follows. Section 2 introduces a reduction to *canonical form*, which is used to determine simple criteria that identify degenerate (linear or planar) RRMF curves, allowing the subsequent analysis to focus on proper RRMF curves, i.e., true space curves. Section 3 further exploits the canonical form reduction to facilitate the enumeration of all possible proper PH quintic solutions to the RRMF condition. In addition to the known class of RRMF quintics satisfying (10) with $\deg(a^2 + b^2) = 4$, this enumeration reveals the existence of a novel class of proper RRMF quintics satisfying (10) with $\deg(a^2 + b^2) = 2$, having the same number of freedoms as the previously–known solutions. Finally, Section 4 considers the "inverse" problem of generating RRMF curves from quadruples $u(t), v(t), p(t), q(t)$ obtained from the (infinitely many) decompositions $f(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t)$ of any given strictly positive real polynomial $f(t)$, while Section 5 summarizes the results of this paper, and identifies open problems for further investigation.

2 Degenerate PH curves

Since every straight line and every planar PH curve is trivially an RRMF curve, and we are interested in true space curves, instances of (3) or (5) that define straight lines or planar curves will be called *degenerate* spatial PH curves. We present here new criteria to identify such degenerate curves, based on Lemma 1 below and the fact that two real polynomials $f(t), g(t)$ are linearly dependent if and only if they satisfy $fg' = f'g$. These criteria are independent of non–essential coefficients, and are easy to test in practice.

As in earlier studies [5], the analysis can be greatly simplified by invoking a scaling/rotation transformation to eliminate non–essential freedoms that do not influence the intrinsic nature of a spatial PH curve. We call this transformation *reduction to canonical form.*

Lemma 1. *Let* $\mathbf{r}(t)$ *be a PH curve whose hodograph* $\mathbf{r}'(t)$ *is generated through* (5) *by* $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\, v(t)$ *and* $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}\, p(t)$*, where* $u(t), v(t), p(t), q(t)$ *are polynomials of degree* $m \geq 1$*. Then complex values* $\boldsymbol{\mu}, \boldsymbol{\nu}$ *can be chosen such that, under the transformation*

$$\left[\begin{array}{c} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{array}\right] \to \left[\begin{array}{cc} \boldsymbol{\mu} & -\overline{\boldsymbol{\nu}} \\ \boldsymbol{\nu} & \overline{\boldsymbol{\mu}} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{array}\right], \tag{12}$$

the polynomials $v(t)$*,* $p(t)$*,* $q(t)$ *are of degree* $m - 1$ *at most.*

Proof : If we write $\boldsymbol{\alpha}(t) = \mathbf{a}_m t^m + \cdots + \mathbf{a}_1 t + \mathbf{a}_0$ and $\boldsymbol{\beta}(t) = \mathbf{b}_m t^m + \cdots + \mathbf{b}_1 t + \mathbf{b}_0$ (where $\mathbf{a}_k = u_k + \mathrm{i}\, v_k$ and $\mathbf{b}_k = q_k + \mathrm{i}\, p_k$ for $k = 0, \ldots, m$) the coefficients transform according to

$$\left[\begin{array}{c} \mathbf{a}_k \\ \mathbf{b}_k \end{array}\right] \to \left[\begin{array}{cc} \boldsymbol{\mu} & -\overline{\boldsymbol{\nu}} \\ \boldsymbol{\nu} & \overline{\boldsymbol{\mu}} \end{array}\right] \left[\begin{array}{c} \mathbf{a}_k \\ \mathbf{b}_k \end{array}\right]$$

for $k = 0, \ldots, m$. In particular, with the choices $\boldsymbol{\mu} = \overline{\mathbf{a}}_m / (|\mathbf{a}_m|^2 + |\mathbf{b}_m|^2)$ and $\boldsymbol{\nu} = -\,\mathbf{b}_m / (|\mathbf{a}_m|^2 + |\mathbf{b}_m|^2)$ we obtain $(\mathbf{a}_m, \mathbf{b}_m) \to (1, 0)$. ∎

As noted in [5], the map (12) defines a scaling/rotation in $\mathbb{R}^3$ of the hodograph specified by (5), that does not alter its intrinsic nature. From Lemma 1 we may henceforth assume, without loss of generality, that $u(t) = t^m + \cdots + u_1 t + u_0$ while $v(t), p(t), q(t)$ are of degree $m-1$ at most. We call a quadruple of polynomials $(u(t), v(t), p(t), q(t))$ of this form *canonical.*

Now $\mathbf{r}(t)$ is planar if and only if $x'(t), y'(t), z'(t)$ are linearly dependent. Then we observe from (3) that, in canonical form, $x'(t)$ is of degree $2m$, while $y'(t), z'(t)$ are of degree $2m-1$ at most. Therefore, $\mathbf{r}(t)$ is planar if and only if $y'(t)$ and $z'(t)$ are linearly dependent, i.e., $y'z'' = y''z'$, which is equivalent to

$$(p^2 + q^2)(uv' - u'v) \;=\; (u^2 + v^2)(qp' - q'p)\,. \tag{13}$$

Furthermore, $\mathbf{r}(t)$ is a straight line if and only if $p(t) = q(t) = 0$. Indeed, when $\mathbf{r}(t)$ is a line $x'(t), y'(t)$ and $x'(t), z'(t)$ are linearly dependent, respectively. But since $x'(t)$ is of degree $2m$ and $y'(t), z'(t)$ are of degree $2m-1$ at most, we must have $y'(t) = z'(t) = 0$, and this implies that $p(t) = q(t) = 0$, since $u^2(t) + v^2(t) \neq 0$. The converse is trivial. These results may be summarized as follows.

Proposition 1. *Let* $\mathbf{r}(t)$ *be a PH curve with hodograph defined by the canonical quadruple* $(u(t), v(t), p(t), q(t))$ *as above. Then*

1. $\mathbf{r}(t)$ *is a plane curve, other than a straight line, if and only if* (13) *is satisfied with* $(p(t), q(t)) \neq (0, 0)$*.*
2. *On the other hand,* $\mathbf{r}(t)$ *is a straight line if and only if* $(p(t), q(t)) = (0, 0)$*.*

In canonical–form, a *degenerate* RRMF curve is either a straight line or planar curve that satisfies (13) and has vanishing torsion, while a *proper* RRMF curve is a true space curve that does not satisfy (13) and thus has non–vanishing torsion.

3 Classification of RRMF quintics

In previous studies [5, 6, 9] the RRMF quintics have been studied under the assumption that (10) and (11) are satisfied with $u^2 + v^2 + p^2 + q^2 = a^2 + b^2$ and $|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2 = |\mathbf{w}|^2$, respectively. Also, it was shown in [9] that $\deg(a^2 + b^2) \leq \deg(u^2 + v^2 + p^2 + q^2)$ is a necessary condition for satisfaction of (10). Thus, for RRMF quintics with $\deg(u, v, p, q) = 2$, the possible solutions to (10) may have (i) $\deg(a, b) = 0$, (ii) $\deg(a, b) = 1$, or (iii) $\deg(a, b) = 2$. It will be shown below that case (i) defines only planar curves, while case (iii) has been thoroughly analyzed before [5, 9]. The significant new outcome of this analysis is case (ii), which reveals the existence of a novel class of RRMF quintics that are true space curves.

3.1 Class I RRMF quintics

The spatial PH quintic curves that belong to this class correspond to the case where $a(t), b(t)$ are assumed to be quadratic — i.e., case (iii) above — and hence

$$uv' - u'v - pq' + p'q \;=\; \gamma(ab' - a'b) \qquad \text{and} \qquad u^2 + v^2 + p^2 + q^2 \;=\; \gamma(a^2 + b^2)$$

for some non–zero constant γ (one may, without loss of generality, set $\gamma = 1$). It was shown in [5] that, if the quadratic quaternion polynomial (2) or complex polynomials (4) are specified in the Bernstein basis

$$b_k^m(t) \,=\, \binom{m}{k}(1-t)^{m-k}t^k\,, \quad k=0,\ldots,m\,,$$

on $t \in [\,0,1\,]$ as

$$\mathcal{A}(t) \,=\, \mathcal{A}_0\, b_0^2(t) \,+\, \mathcal{A}_1\, b_1^2(t) \,+\, \mathcal{A}_2\, b_2^2(t)\,, \tag{14}$$

or

$$\boldsymbol{\alpha}(t) \,=\, \boldsymbol{\alpha}_0\, b_0^2(t) \,+\, \boldsymbol{\alpha}_1\, b_1^2(t) \,+\, \boldsymbol{\alpha}_2\, b_2^2(t)\,, \qquad \boldsymbol{\beta}(t) \,=\, \boldsymbol{\beta}_0\, b_0^2(t) \,+\, \boldsymbol{\beta}_1\, b_1^2(t) \,+\, \boldsymbol{\beta}_2\, b_2^2(t)\,, \tag{15}$$

then imposition of the coefficient constraints

$$\mathrm{vect}(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*) \,=\, \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^* \tag{16}$$

or

$$\mathrm{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2 - \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2) \,=\, |\boldsymbol{\alpha}_1|^2 - |\boldsymbol{\beta}_1|^2 \qquad \text{and} \qquad \boldsymbol{\alpha}_0\overline{\boldsymbol{\beta}}_2 + \boldsymbol{\alpha}_2\overline{\boldsymbol{\beta}}_0 \,=\, 2\,\boldsymbol{\alpha}_1\overline{\boldsymbol{\beta}}_1 \tag{17}$$

is sufficient and necessary for an RRMF curve satisfying (10) with $a(t), b(t)$ quadratic, or equivalently (11) with $\mathbf{w}(t)$ quadratic. Since these Class I RRMF quintics have been thoroughly analyzed before [5, 9] we shall not dwell further on them here.

We henceforth focus on PH quintics satisfying (10) and (11) with $\deg(a^2+b^2) < \deg(u^2+v^2+p^2+q^2)$ and $\deg(|\mathbf{w}|^2) < \deg(|\boldsymbol{\alpha}|^2+|\boldsymbol{\beta}|^2)$. As noted above, there are only two possible cases — (i) $\deg(a^2+b^2)=0$, and (ii) $\deg(a^2+b^2)=2$.

3.2 Class II RRMF quintics

We begin with the observation that a scaling/rotation transformation does not influence the RRMF nature of a spatial PH curve.

Lemma 2. *If the RRMF condition* (11) *is satisfied by complex polynomials* $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ *and* $\mathbf{w}(t)$, *it is also satisfied upon replacing them by* $\boldsymbol{\mu}\,\boldsymbol{\alpha}(t) - \overline{\boldsymbol{\nu}}\,\boldsymbol{\beta}(t)$, $\boldsymbol{\nu}\,\boldsymbol{\alpha}(t) + \overline{\boldsymbol{\mu}}\,\boldsymbol{\beta}(t)$ *and* $\boldsymbol{\eta}\,\mathbf{w}(t)$, *for any complex numbers* $(\boldsymbol{\mu},\boldsymbol{\nu}) \neq (0,0)$ *and* $\boldsymbol{\eta} \neq 0$.

Proof : For complex numbers $(\boldsymbol{\mu},\boldsymbol{\nu}) \neq (0,0)$ the linear map (12) applied to the polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ yields

$$\begin{aligned} |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2 \;&\to\; (|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)\,(|\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2)\,, \\ \overline{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t) + \overline{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t) \;&\to\; (|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)\,(\overline{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t) + \overline{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t))\,, \end{aligned}$$

and hence the left–hand side of (11) is unaltered. Similarly, we have $\mathrm{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t)) \to |\boldsymbol{\eta}|^2\,\mathrm{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t))$ and $|\mathbf{w}(t)|^2 \to |\boldsymbol{\eta}|^2\,|\mathbf{w}(t)|^2$ when $\mathbf{w}(t) \to \boldsymbol{\eta}\,\mathbf{w}(t)$, so the the right–hand side of (11) is also unchanged. ■

Now from Lemma 1 we may henceforth assume, without loss of generality, that

$$u(t) \,=\, t^2 + u_1 t + u_0\,, \quad v(t) \,=\, v_1 t + v_0\,, \quad p(t) \,=\, p_1 t + p_0\,, \quad q(t) = q_1 t + q_0\,. \tag{18}$$

We are primarily concerned here with case (ii), but before addressing it we quickly dispense with case (i).

Case (i): $\deg(a^2+b^2)=0$. Since $ab'-a'b=0$, we deduce from (10) and (18) that $uv'-u'v-pq'+p'q = -v_1t^2 - 2v_0t + u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0 = 0$, so we must have $v_1 = v_0 = 0$ and thus $v(t)=0$. But then condition (13) is satisfied, i.e., the curve is planar and is thus a degenerate RRMF curve.

We focus henceforth on case (ii), which yields non–degenerate RRMF curves — i.e., true spatial PH quintics with rational rotation–minimizing frames. It transpires that, in canonical form, this class of RRMF quintics exhibits five free parameters — as with the case (iii) RRMF quintics satisfying (10) with $\deg(a^2+b^2)=4$. In prior studies [5, 6, 9] these latter curves were called "generic" RRMF quintics, in the expectation that solutions to (10) with $\deg(u^2+v^2+p^2+q^2)=4$ and $\deg(a^2+b^2)<4$ would comprise a "lower–dimension subspace" of the complete set of non–degenerate RRMF quintics. Since this is not the case, we shall refer to non–degenerate RRMF quintics that satisfy (10) with $\deg(a^2+b^2)=4$ and $\deg(a^2+b^2)=2$ as *Class I* and *Class II*, respectively. We now proceed with the analysis of the latter curves.

Case (ii): $\deg(a^2+b^2)=2$. Since $a(t)$, $b(t)$ are linear and relatively prime, Lemma 2 indicates that we may, without loss of generality, assume $a(t)=t-r$, $b(t)=s$ for $r,s\in\mathbb{R}$ with $s\neq 0$. Let $w=uv'-u'v-pq'+p'q$ and $\sigma=u^2+v^2+p^2+q^2$. Then (10) implies that

$$[\,(t-r)^2+s^2\,]\,w(t)\;=-\,s\,\sigma(t)\,. \tag{19}$$

Now from (18) we have

$$\begin{aligned} w(t) &= -\,v_1t^2-2v_0t+u_0v_1-u_1v_0-p_0q_1+p_1q_0\,,\\ \sigma(t) &= t^4\,+\,2u_1t^3\,+\,(2u_0+u_1^2+v_1^2+p_1^2+q_1^2)t^2\\ &\quad +\,2(u_0u_1+v_0v_1+p_0p_1+q_0q_1)t\,+\,u_0^2+v_0^2+p_0^2+q_0^2\,, \end{aligned} \tag{20}$$

and comparing like powers of t on the left and right in (19) yields

$$\begin{aligned} s-v_1 &= 0\,,\\ 2u_1s+2v_1r-2v_0 &= 0\,,\\ (2u_0+u_1^2+v_1^2+p_1^2+q_1^2)s+u_0v_1-u_1v_0-p_0q_1+p_1q_0-v_1(r^2+s^2)+4v_0r &= 0\,,\\ 2(u_0u_1+v_0v_1+p_0p_1+q_0q_1)s-2(u_0v_1-u_1v_0-p_0q_1+p_1q_0)r-2v_0(r^2+s^2) &= 0\,,\\ (u_0^2+v_0^2+p_0^2+q_0^2)s+(u_0v_1-u_1v_0-p_0q_1+p_1q_0)(r^2+s^2) &= 0\,. \end{aligned} \tag{21}$$

This is a system of five equations in ten variables, and its solutions may be characterized as follows.

Proposition 2. *The (real) solutions of the system* (21) *can be parameterized in terms of the free variables* r,u_1,v_1,p_1,q_1 *with* $v_1\neq 0$*, as* $s=v_1$ *and either*

$$u_0\,=\,-(u_1+r)r\,,\quad v_0\,=\,(u_1+r)v_1\,,\quad p_0\,=\,v_1q_1-p_1r\,,\quad q_0\,=\,-(v_1p_1+q_1r)\,, \tag{22}$$

or

$$\begin{aligned} u_0 &= -\,(u_1+r)r\,-\,\frac{4v_1^2(p_1^2+q_1^2)}{(u_1+2r)^2+9v_1^2+p_1^2+q_1^2}\,,\\ v_0 &= (u_1+r)v_1\,,\\ p_0 &= v_1q_1-p_1r\,+\,\frac{4v_1^2((u_1+2r)p_1-3v_1q_1)}{(u_1+2r)^2+9v_1^2+p_1^2+q_1^2}\,,\\ q_0 &= -(v_1p_1+q_1r)\,+\,\frac{4v_1^2((u_1+2r)q_1+3v_1p_1)}{(u_1+2r)^2+9v_1^2+p_1^2+q_1^2}\,. \end{aligned} \tag{23}$$

Proof : We first substitute $s=v_1$ and $v_0=(u_1+r)v_1$ from the first two of equations (21) into the remaining three. From the third and fourth equations we then obtain

$$(p_1^2+q_1^2)\,p_0\;=\;f_0 \qquad\text{and}\qquad (p_1^2+q_1^2)\,q_0\;=\;g_0\,,$$

where f_0,g_0 are polynomials in u_0,u_1,v_1,p_1,q_1,r. Assume for now that $p_1^2+q_1^2\neq 0$. Substituting for p_0,q_0 from the above into the fifth of equations (21) and solving for u_0, we obtain the first expression in either (22) or in (23).

Substituting for u_0 from (22) into the third and fourth of equations (21) then yields $p_0=v_1q_1-p_1r$ and $q_0=-v_1p_1-q_1r$, thus completing the solution (22). On the other hand, substituting from (23) for u_0 we obtain (through a judicious re–arrangement of terms) the expressions for p_0,q_0 given in (23).

Suppose now that $p_1^2+q_1^2=0$. In that case, the third equation gives $u_0=-(u_1+r)r$ and on substituting this and $s=v_1$, $v_0=(u_1+r)v_1$ into the fifth equation we obtain $(p_0^2+q_0^2)v_1=0$. Since $v_1\neq 0$, we see that $p_0=q_0=0$ as required by solution (22). Finally, note that the denominator in the expressions for u_0,p_0,q_0 in (23) is never zero, since by assumption $v_1\neq 0$. This concludes the proof. ■

For curves defined by the solution (22), we have

$$\begin{aligned} x'(t) &= [\,(t-r)^2+v_1^2\,]\,[\,(t+u_1+r)^2-p_1^2-q_1^2\,]\,,\\ y'(t) &= 2q_1\,[\,(t-r)^2+v_1^2\,]\,(t+u_1+r)\,,\\ z'(t) &= -\,2p_1\,[\,(t-r)^2+v_1^2\,]\,(t+u_1+r)\,. \end{aligned}$$

Such curves are evidently *planar* and *non–primitive*, since $x'(t), y'(t), z'(t)$ are linearly dependent and have a non–constant common factor. The curves defined by the solution (23), on the other hand, are primitive and are true space curves with $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' \neq 0$.

Defining the complex numbers $\mathbf{a}_k = u_k + \mathrm{i}\, v_k$ and $\mathbf{b}_k = q_k + \mathrm{i}\, p_k$ for $k = 0, 1, 2$ and writing $\boldsymbol{\gamma} = -\,r + v_1\mathrm{i}$, $\boldsymbol{\zeta} = \mathbf{a}_1 - 2\,\overline{\boldsymbol{\gamma}}$, $\boldsymbol{\eta} = \mathbf{b}_1$, the solutions (23) that define proper Class II RRMF quintics can be more compactly expressed as

$$\mathbf{a}_0 \,=\, (\mathbf{a}_1 - \boldsymbol{\gamma})\boldsymbol{\gamma} \,-\, \frac{4v_1^2|\boldsymbol{\eta}|^2}{|\boldsymbol{\zeta}|^2 + |\boldsymbol{\eta}|^2}\,, \qquad \mathbf{b}_0 \,=\, \boldsymbol{\gamma}\,\boldsymbol{\eta} \,+\, \frac{4v_1^2\overline{\boldsymbol{\zeta}}\boldsymbol{\eta}}{|\boldsymbol{\zeta}|^2 + |\boldsymbol{\eta}|^2}\,. \tag{24}$$

Since they provide a means of generating Class II RRMF quintics in terms of one real and two complex free parameters — r and $\mathbf{a}_1$, $\mathbf{b}_1$ — the relations (24) might be considered analogous to the generating formulae for Class I RRMF quintics specified in Proposition 1 of [5]. However, they are obviously more complicated than equations (15) in [5], and have thus far eluded a reduction to simple sufficient–and–necessary coefficient constraints, analogous to (16) and (17) for Class I curves. This problem deserves further attention, but at present it seems clear that Class I RRMF quintics have a simpler algebraic structure than Class II.

Remark 1. On a PH quintic, the ERF vectors (8) are quartic rational functions of the curve parameter. For Class I RRMF quintics, satisfying (10) with $\deg(a(t), b(t)) = 2$, the RMF normal–plane vectors defined by (9) are rational functions of degree 8. Since the solution (23) identifies RRMF quintics satisfying (10) with $\deg(a(t), b(t)) = 1$, the RMF vectors (9) on these Class II RRMF quintics are only of degree 6.

Example 1. Choosing the values $r = 1$, $u_1 = -1$, $v_1 = 2$, $p_1 = 0$, $q_1 = -2$ in (23) gives

$$s = 2\,, \quad u_0 = -\frac{64}{41}\,, \quad v_0 = 0\,, \quad p_0 = \frac{28}{41}\,, \quad q_0 = \frac{50}{41}\,,$$

and hence we have

$$u(t) = t^2 - t - \frac{64}{41}\,, \quad v(t) = 2\,t\,, \quad p(t) = \frac{28}{41}\,, \quad q(t) = -\,2\,t + \frac{50}{41}$$

and

$$a(t) = t - 1\,, \quad b(t) = 2\,,$$

which satisfy

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} \,=\, \frac{ab' - a'b}{a^2 + b^2} \,=\, \frac{-\,2}{t^2 - 2\,t + 5}\,.$$

The resulting hodograph components

$$\begin{aligned}
x'(t) \;&=\; u^2(t) + v^2(t) - p^2(t) - q^2(t) \;=\; t^4 \,-\, 2\,t^3 \,-\, \frac{87}{41}\,t^2 \,+\, 8\,t \,+\, \frac{812}{1681}\,, \\
y'(t) \;&=\; 2\,[\,u(t)q(t) + v(t)p(t)\,] \;=\; -\,4\,t^3 \,+\, \frac{264}{41}\,t^2 \,+\, \frac{268}{41}\,t \,-\, \frac{6400}{1681}\,, \\
z'(t) \;&=\; 2\,[\,v(t)q(t) - u(t)p(t)\,] \;=\; -\,\frac{384}{41}\,t^2 \,+\, \frac{256}{41}\,t \,+\, \frac{3584}{1681}\,,
\end{aligned}$$

define a primitive curve with $\gcd(x'(t), y'(t), z'(t)) = 1$ and they satisfy $x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t)$, where

$$\sigma(t) \,=\, (t^2 - 2\,t + 5)\left(t^2 + \frac{36}{41}\right)\,.$$

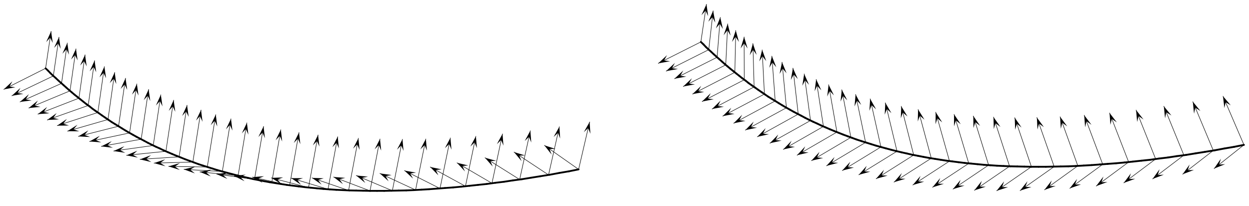

Figure 1: The RRMF quintic $\mathbf{r}(t)$ of Example 1 for $t \in [\,1.5, 2.5\,]$, showing the variation of the principal normal and binormal vectors of the Frenet frame (left), and the rational RMF normal–plane vectors along the curve (right). The initial RMF orientation is chosen to agree with the Frenet frame at the left point.

The hodograph defines a true space curve, as can be verified from the fact that condition (13) is not satisfied, and $\mathbf{r}(t)$ has the non–constant torsion

$$\tau \,=\, \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} \,=\, \frac{-\,7872}{(t^2-2\,t+5)^2(1681\,t^2-738\,t+2997)}\,,$$

while the curvature is given by

$$\kappa \,=\, \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\,\mathbf{r}'\,|^3} \,=\, \frac{164\sqrt{(t^2-2\,t+5)(1681\,t^2-738\,t+2997)}}{(t^2-2\,t+5)^2(41\,t^2+36)^2}\,.$$

The rational RMF normal–plane vectors, obtained from (8) and (9), are of degree 6 in the curve parameter t. Figure 1 compares these vectors with the Frenet frame normal–plane vectors, over the interval $t\in[\,1.5,2.5\,]$. The angular speed of the Frenet frame and the rational RMF are compared over the same interval in Figure 2.

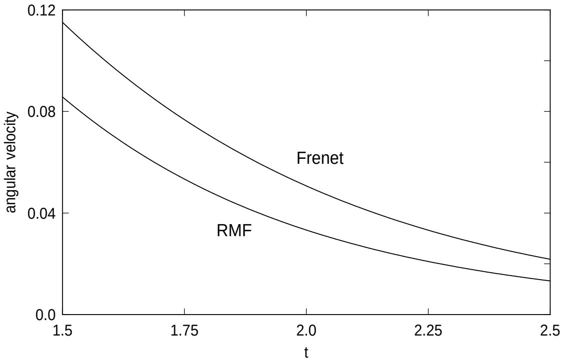

Figure 2: Comparison of angular speed for the Frenet frame and the rational RMF along the RRMF quintic of Example 1, over the parameter interval $t\in[\,1.5,2.5\,]$ illustrated in Figure 1.

Example 2. Choosing the values $r=2$, $u_1=-2$, $v_1=1$, $p_1=2$, $q_1=1$ in (23) gives

$$s=1\,,\quad u_0=-\frac{10}{9}\,,\quad v_0=0\,,\quad p_0=-\frac{25}{9}\,,\quad q_0=-\frac{20}{9}\,,$$

and thus we obtain

$$u(t)=t^2-2\,t-\frac{10}{9}\,,\quad v(t)=t\,,\quad p(t)=2t-\frac{25}{9}\,,\quad q(t)=t-\frac{20}{9}$$

and

$$a(t)=t-2\,,\quad b(t)=1\,,$$

so that

$$\frac{uv'-u'v-pq'+p'q}{u^2+v^2+p^2+q^2} \,=\, \frac{ab'-a'b}{a^2+b^2} \,=\, \frac{-\,1}{t^2-4\,t+5}\,.$$

The resulting hodograph components

$$\begin{aligned} x'(t) &= u^2(t)+v^2(t)-p^2(t)-q^2(t) \,=\, t^4 \,-\, 4\,t^3 \,-\, \frac{20}{9}\,t^2 \,+\, 20\,t \,+\, \frac{925}{81}\,, \\ y'(t) &= 2\,[\,u(t)q(t)+v(t)p(t)\,] \,=\, 2\,t^3 \,-\, \frac{40}{9}\,t^2 \,+\, \frac{10}{9}\,t \,+\, \frac{400}{81}\,, \\ z'(t) &= 2\,[\,v(t)q(t)-u(t)p(t)\,] \,=\, -\,4\,t^3 \,+\, \frac{140}{9}\,t^2 \,-\, \frac{100}{9}\,t \,-\, \frac{500}{81}\,, \end{aligned}$$

define a primitive curve with $\gcd(x'(t),y'(t),z'(t))=1$ and they satisfy $x'^2(t)+y'^2(t)+z'^2(t)=\sigma^2(t)$, where

$$\sigma(t) \,=\, (t^2-4\,t+5)\left(t^2+\frac{25}{9}\right)\,.$$

Again, this example defines a true space curve, with the curvature and torsion functions

$$\kappa \,=\, \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\,\mathbf{r}'\,|^3} \,=\, \frac{18\sqrt{5\,(t^2-4\,t+5)(81\,t^2-180\,t+325)}}{(t^2-4\,t+5)^2(9\,t^2+25)^2}\,,$$

$$\tau \,=\, \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} \,=\, \frac{-\,108}{(t^2-4\,t+5)^2(81\,t^2-180\,t+325)}\,.$$

Finally, although case (iii) with $\deg(a^2+b^2)=4$ has been thoroughly treated in [5, 6], for completeness we briefly consider the special instance of this case where $\gcd(w,\sigma)$ is of degree 2, i.e., $w(t)$ is a factor of $\sigma(t)$.

Remark 2. Let $\mathbf{r}(t)$ be an RRMF quintic defined by the polynomials (18), such that (10) is satisfied with $\deg(a^2+b^2)=4$ and $\gcd(w,\sigma)$ is of degree 2. Then $p(t)=q(t)=0$.

Proof : In this case, we must have $w=ab'-a'b$ and $\sigma=a^2+b^2$, so that $\gcd(ab'-a'b,a^2+b^2)$ is also of degree 2. Since $\gcd(ab'-a'b,a^2+b^2)=\gcd(2(aa'+bb'),a^2+b^2)$ by Lemma 4.1 in [9], a^2+b^2 must have complex conjugate roots $\mathbf{z},\overline{\mathbf{z}}=r\pm \mathrm{i}\,s$ (where $r,s\in\mathbb{R}$ and $s\neq 0$) of multiplicity 2 each. Therefore,

$$\sigma(t) \,=\, a^2(t)+b^2(t) \,=\, [\,(t-r)^2+s^2\,]^2\,. \tag{25}$$

Also, $(ab'-a'b)/(a^2+b^2)$ must have the form given in equation (12) of [9], namely

$$\frac{ab'-a'b}{a^2+b^2} \,=\, \pm\frac{\mathrm{i}}{2}\left[\frac{2}{t-\mathbf{z}}-\frac{2}{t-\overline{\mathbf{z}}}\right].$$

Consequently, $w=ab'-a'b$ may be written as

$$w(t) \,=\, \pm\, 2s\,[\,(t-r)^2+s^2\,]\,. \tag{26}$$

Choosing the $-$ sign above and comparing (25)–(26) with (20) we obtain, after simplification, the equations

$$\begin{aligned}
v_1-2s \;&=\; 0\,,\\
v_0+2rs \;&=\; 0\,,\\
u_1+2r \;&=\; 0\,,\\
2su_0-p_0q_1+p_1q_0-2r^2s+2s^3 \;&=\; 0\,,\\
2u_0+p_1^2+q_1^2-2r^2+2s^2 \;&=\; 0\,,\\
2ru_0-p_0p_1-q_0q_1-2r^3+2rs^2 \;&=\; 0\,,\\
u_0^2+p_0^2+q_0^2-r^4+2r^2s^2-s^4 \;&=\; 0\,.
\end{aligned}$$

We claim that $p_1^2+q_1^2=0$. Assume the contrary. Then, solving the fourth and sixth equations for p_0,q_0 and taking into account the fifth equation, we obtain $p_0=-(sq_1+rp_1)$, $q_0=sp_1-rq_1$.

Substituting for p_0,q_0 into the seventh equation, and using the fifth equation again, gives the quadratic

$$u_0^2-2(r^2+s^2)u_0+r^4+2r^2s^2-3s^4 \,=\, 0$$

for u_0, with solutions $u_0=r^2-s^2$ and $u_0=r^2+3s^2$. However, since both solutions contradict the fifth equation, we must have $p_1=q_1=0$. In that case, the fifth and seventh equations give $p_0=q_0=0$, so that $p(t)=q(t)=0$, as claimed. Clearly, $\mathbf{r}(t)$ is then just a straight line (the x–axis). ∎

4 Inverse problem for RRMF quintics

Thus far, we have investigated the conditions on a quaternion polynomial $\mathcal{A}(t)=u(t)+v(t)\,\mathbf{i}+p(t)\,\mathbf{j}+q(t)\,\mathbf{k}$ (assumed primitive, i.e., $\gcd(u(t),v(t),p(t),q(t))=1$) that ensure satisfaction of the RRMF condition (10). Note that, since $\mathcal{A}(t)$ is primitive, the polynomial $|\mathcal{A}(t)|^2=u^2(t)+v^2(t)+p^2(t)+q^2(t)$ is a *positive* real polynomial of even degree.

Now it is well–known that any positive polynomial $f(t)$ of degree $2m$ can be expressed as a sum of squares of four polynomials,

$$f(t) \,=\, u^2(t)+v^2(t)+p^2(t)+q^2(t)\,,$$

in infinitely many ways. We say that $f(t)$ *generates* an RRMF curve $\mathbf{r}(t)$ if a primitive quaternion polynomial $\mathcal{A}(t)=u(t)+v(t)\,\mathbf{i}+p(t)\,\mathbf{j}+q(t)\,\mathbf{k}$ exists, such that $\mathbf{r}'(t)=\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$ and $|\mathcal{A}(t)|^2=f(t)$. It seems natural, then, to pose the following question.

Question 1. Does a given positive real polynomial $f(t)$ of degree $2m$ generate any RRMF curves? If so, are they *proper* or *degenerate* (see Section 2)? Moreover, what are the necessary conditions (if any) on $f(t)$, and its root structure?

We now address this question in the particular case of Class I RRMF curves, when $f(t)$ square–free and $m = 2$. The complete analysis of this problem is deferred to a future study.

Proposition 3. *Let $f(t)$ be a given positive square–free polynomial, of degree 4. Then $f(t)$ always generates straight lines, but only generates planar PH curves and proper RRMF space curves conditionally.*

Proof. Let $f(t) = t^4 + f_3t^3 + f_2t^2 + f_1t + f_0$. Through a change of variables $t \to t - \frac{1}{4}f_3$ we may assume that $f_3 = 0$ without loss of generality. Also, let $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ be a *primitive* quaternion polynomial (in canonical form) defining the hodograph of an RRMF curve $\mathbf{r}(t)$ generated by $f(t)$. Since $f_3 = 0$, we must have $u_1 = 0$ in (18) and thus $u(t) = t^2 + u_0$, $v(t) = v_1t + v_0$, $p(t) = p_1t + p_0$, $q(t) = q_1t + q_0$.

Now since $f(t)$ is square–free, it must have the form $f(t) = [\,(t-r)^2 + k^2\,]\,[\,(t+r)^2 + l^2\,]$ for $r, k, l \in \mathbb{R}$ with $k\,l \neq 0$ and $r^2 + (k+l)^2 > 0$. Also notice that the polynomials $a(t)$, $b(t)$ in (10) must be of the form $a(t) = t^2 + a_0$, $b(t) = b_1t + b_0$. Equating coefficients of like powers of t for the numerators and denominators in (10) then gives $(b_1, b_0) = (v_1, v_0)$ and

$$u_0v_1 - p_0q_1 + p_1q_0 \;=\; a_0v_1\,, \quad 2u_0 + p_1^2 + q_1^2 \;=\; 2a_0\,, \quad p_0p_1 + q_0q_1 \;=\; 0\,, \quad u_0^2 + p_0^2 + q_0^2 \;=\; a_0^2\,. \tag{27}$$

The second and fourth of these equations imply that, if $p_1 = q_1 = 0$, then $p_0 = q_0 = 0$ as well. Furthermore, eliminating a_0 between the first and second equations yields

$$p_1q_0 - p_0q_1 \;=\; \tfrac{1}{2}(p_1^2 + q_1^2)v_1\,, \tag{28}$$

and solving this together with the third equations — under the assumption that $p_1^2 + q_1^2 \neq 0$ — gives

$$p_0 \;=\; -\,\tfrac{1}{2}v_1q_1 \quad \text{and} \quad q_0 \;=\; \tfrac{1}{2}v_1p_1\,. \tag{29}$$

Now if $u(t) = t^2 - r^2 - kl$, $v(t) = l(t-r) + k(t+r)$, $p(t) = 0$, $q(t) = 0$, then $f(t) = u^2(t) + v^2(t)$ and $\gcd(u, v) = 1$, since $k\,l \neq 0$ and $r^2 + (k+l)^2 > 0$. Proposition 1 then indicates that $\mathbf{r}(t)$ is a straight line.

Suppose now that $f(t)$ generates a planar PH curve, satisfying (13) with $(p(t), q(t)) \neq (0, 0)$. From the above arguments we must have $p_1^2 + q_1^2 > 0$, and hence relations (29) hold. Equating coefficients of t^4 on the left and right in (13) and using (28) then gives $-(p_1^2 + q_1^2)v_1 = \frac{1}{2}(p_1^2 + q_1^2)v_1$, and hence $v_1 = 0$. Therefore, $p_0 = q_0 = 0$ from (29), so $qp' - q'p = 0$ and (13) implies that $uv' - u'v = -\,2v_0t = 0$, and we thus have $u(t) = t^2 + u_0$, $v(t) = 0$, $p(t) = p_1t$, $q(t) = q_1t$. Now since the coefficients of t in f and in $u^2 + v^2 + p^2 + q^2$ are $2r(k^2 - l^2)$ and 0, respectively, we must also have either $r = 0$ or $k^2 - l^2 = 0$. When $r = 0$ we obtain $p_1^2 + q_1^2 = (k \pm l)^2$, while if $k^2 = l^2$ we have $p_1^2 + q_1^2 = 4k^2$, and both these cases yield planar PH curves.

Finally, suppose that $f(t)$ generates a proper RRMF space curve. Then, from expression (12) in [9], condition (10) takes the form

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} \;=\; \pm\,\frac{k}{(t-r)^2 + k^2} \;\pm\; \frac{l}{(t+r)^2 + l^2}\,. \tag{30}$$

Now substituting $a_0 = u_0 + \frac{1}{2}(p_1^2 + q_1^2)$ from the second equation in (27) into the fourth equation, we obtain $4u_0 = v_1^2 - (p_1^2 + q_1^2)$, and comparing coefficients of the t^2 term in the numerators in (30) gives $v_1^2 = (k \pm l)^2$. Substituting these results into the equation

$$2u_0 + v_1^2 + p_1^2 + q_1^2 \;=\; k^2 + l^2 - r^2$$

obtained by equating coefficients of t^2 in the denominators in (30), we have $p_1^2 + q_1^2 = -\,(4r^2 + k^2 + l^2 \pm 6kl)$. Therefore, $f(t)$ generates a proper RRMF space curve if and only if $4r^2 + k^2 + k^2 \pm 6kl < 0$. ∎

Example 3. *Let $f(t) = t^4 - 3t^2 - 12t + 40$, with roots $2 \pm i$ and $-2 \pm 2i$. Then the only Class I RRMF curves that $f(t)$ generates are straight lines.*

Proof. With $r = 2$, $k = 1$, $l = 2$ we note that $r(k^2 - l^2) = -6 \neq 0$ and $4r^2 + k^2 + l^2 \pm 6kl = 33$ or $9 > 0$. ∎

5 Closure

A complete characterization of all quintic curves with rational rotation–minimizing frames (RRMF quintics) has been developed, through their reduction to canonical form by a spatial scaling/rotation transformation. The characterization incorporates a succinct identification of *degenerate* solutions (straight lines and planar PH curves) through the condition (13). For *proper* RRMF quintics (true space curves) a new set of solutions, the Class II RRMF quintics, has been identified satisfying the RRMF condition (10) with $\deg(a(t), b(t)) = 1$, as distinct from the previously–known Class I RRMF quintics that satisfy (10) with $\deg(a(t), b(t)) = 2$. As with the Class I RRMF quintics, the Class II RRMF quintics depend upon five free parameters, and their rational RMFs are of somewhat lower degree (six rather than eight).

However, the parameterization (23) of the set of Class II curves is more complicated than the corresponding representation for Class I curves. Concerted efforts to derive simpler generating formulas for Class II curves, or sufficient–and–necessary constraints on the quaternion or Hopf map coefficients of spatial PH curves for a Class II curve (such as are available for Class I curves) have thus far been unsuccessful. This topic deserves further investigation, due to its importance in making these new RRMF curves amenable to the development of algorithms for practical use in animation, spatial path planning, and geometric design.

Finally, a new approach to the RRMF curves has been proposed, based on considering the four polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ in (2) or (4) to be generated by the decomposition of a positive polynomial $f(t)$ as a sum of four squares. Some preliminary results concerning the construction of Class I RRMF quintics through this approach were presented.

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